LARGE $2P_3$-FREE GRAPHS WITH BOUNDED DEGREE

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Abstract

Let $ex^*(D;H)$ be the maximum number of edges in a connected graph with maximum degree D and no induced subgraph H; this is finite if and only if H is a disjoint union of paths. If the largest component of such an H has order m, then $ex^*(D;H) = O(D^2 ex^*(D;P_m))$. Constructively, $ex^*(D;qP_m) = \Theta(qD^2 ex^*(D;P_m))$ if $q > 1$ and $m > 2$ ($\Theta(qD^2)$ if $m = 2$). For $H = 2P_3$ (and $D \geq 8$), the maximum number of edges is $\frac{1}{8}[D^4 + D^3 + D^2 + 6D]$ if D is even and $\frac{1}{8}[D^4 + D^3 + 2D^2 + 3D + 1]$ if D is odd, achieved by a unique extremal graph.

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Running head: LARGE $2P_3$-FREE GRAPHS

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1. INTRODUCTION

The archetypal problem of extremal graph theory is to determine the maximum number of edges in an n-vertex graph that does not contain some fixed graph H as a subgraph; this is usually written as $ex(n;H)$. Turán solved this for cliques; an extensive discussion of this and related problems appears in [1]. In this paper we study large graphs when we forbid H as an *induced* subgraph; such graphs are called *H-free*. Since any large clique is H-free when H is not a clique, and since disjoint copies of a graph introduce no new connected subgraphs, we obtain a more sensible problem by defining $ex^*(D;H)$ to be the maximum number of edges in a connected graph G with maximum degree at most D. The problem has been solved for $H=P_4$ in [3] and for $H=2K_2$ by F. Chung, Gyarfás, Trotter, and Tuza in [2]. The results are $ex^*(D;P_4)=D^2$ for all D, and $ex^*(D;2K_2)=\frac{5}{4}D^2$ when D is even. In these cases the extremal graph is unique; for $H=P_4$ the graph is $K_{D,D}$, and for $H=2K_2$ it is the graph obtained by expanding each vertex of a 5-cycle into $D/2$ vertices with the same neighborhood ($ex^*(D;2K_2)$ is slightly smaller as a function of D when D is odd). In this paper, we solve the problem exactly for $H=2P_3$ except for small values of D; again the extremal graph is unique. We refer to a largest connected H-free graph with maximum degree D as an *extremal H-free graph.*

It was observed in [3] that $ex^*(D;H)$ is undefined unless H is a disjoint union of paths. An arbitrarily long cycle has no fixed-length cycle or vertex of degree more than 2, so there are arbitrarily large 2-regular H-free connected graphs (and also D-regular D-connected H-free graphs [3]) unless H is acyclic and has maximum degree 2.

If $H=\cup_{i=1}^{q}P_{m_i}$ and $m=q-1+\sum_{i=1}^{q}m_i$, then an H-free graph is also P_m-free. The diameter of a P_m-free graph is less than $m-1$. A graph with maximum degree at most D and diameter less than $m-1$ has at most $1+D\frac{(D-1)^{m-2}-1}{D-2}$ vertices (by counting the vertices reachable at each distance from a fixed vertex). Multiplying by $D/2$ to count edges, we have $ex^*(D;P_m)$ bounded by a polynomial in D with leading term $\frac{1}{2}D^{m-1}$ [3]. Simple constructions in [3] achieve roughly the square root of this bound; the P_7-free graph constructed there is in fact the unique largest $2P_3$-free graph.

The upper bounds can be improved recursively when H has more than one component. If the largest component of H has order m, then $ex^*(D;H)=O(D^2ex^*(D;P_m))$. We prove constructively that $ex^*(D;qP_m)=\Theta(qD^2ex^*(D;P_m))$ for $q>1$ ($\Theta(qD^2)$ for $m=2$). we have determined the complete answer for $qP_m=2P_3$. The constructions and recursive approach appear in Section 2, with the upper bound for $ex^*(D;2P_3)$ developed in the remainder of the paper.

Let $n(G)=|V(G)|$ and $e(G)=|E(G)|$ denote the order and size of a graph G. We use $N(x)=\{u\in V(G)\colon ux\in E(G)\}$ for the *neighborhood* of vertex x, and we let $N(S)=\cup_{x\in S}N(x)$. We use $G[S]$ to denote the subgraph of G induced by the vertex subset S. The degree of a vertex in G is $d(v)$, the maximum degree of G is $\Delta(G)$, and the number of vertices in the largest clique in G is $\omega(G)$. We use "clique" to refer both to a complete subgraph and to the vertex set of a complete subgraph, as is common with "independent set" for a clique in the complement.

2. THE GENERAL PROBLEM

Suppose $H = \sum_{i=1}^{q} P_{m_i}$, and let $H' = \sum_{i=1}^{q-1} P_{m_i}$. Because H' is an induced subgraph of H, every H'-free graph is also H-free, and $ex^*(D; H') \le ex^*(D; H)$. If strict inequality holds, the extremal graph must have H' as an induced subgraph. This leads to the bound below.

THEOREM 1. If $H = \sum_{i=1}^{q} P_{m_i}$ and $H' = \sum_{i=1}^{q-1} P_{m_i}$, then

$$ex^*(D; H) \le \max\{ex^*(D; H'), n(H')D^2[ex^*(D; P_{m_q}) + 1]\}.$$

Proof: Let G be an extremal H-free graph. As remarked above, $e(G)$ is bounded by $ex^*(D; H')$ unless G contains an induced copy of H'. Suppose $M_0 \subseteq V(G)$ induces H'. Let $M_1 = N(M_0) - M_0$, and let $M_2 = V(G) - (M_0 \cup M_1)$. Because there is no edge between M_0 and M_2, every component of $G[M_2]$ is P_{m_q}-free. There are at most $|M_1|(D-1)$ such components. Since $|M_0| = n(H')$, we have $|M_1| \le (D-1)n(H')$. Since each edge induced by M_1 contributes only once to $e(G)$, the edge count is bounded by allowing each edge incident to $x \in M_1$ to give rise to such a component, except for one edge to M_0. The resulting bound is $e(G) \le n(H')(D-1)^2[ex^*(D; P_{m_q}) + 1]\}$. □

As remarked earlier, this implies $ex^*(D; H) = O(D^2 ex^*(D; P_m))$ when the largest component of H is P_m, except that for $m = 2$ we start with $2P_2$ since $ex^*(D; P_2) = 0$. We next present a construction to achieve this order of growth (as a function of D) for $H = qP_m$.

THEOREM 2. Letting $p(D, m) = ex^*(D; P_m)$, we have
$(2q-1)\frac{D^2}{12}(p(D-1, m) + 1) \le ex^*(D; qP_m) \le m(q-1)D^2(p(D, m) + 1)$. For $m = 2$, the lower bound can be improved to $(3q-1)D^2/4$.

Proof: For simplicity, assume D is a multiple of 6. Arrange $2q-1$ cliques cyclically, each having $D/6$ vertices. Each vertex is adjacent to every vertex in its own clique and in the two neighboring cliques; this contributes $D/2-1$ to the degree of each vertex. Add $D/2+1$ pendant edges incident to each vertex, with each such edge also being incident to one of $(2q-1)\frac{D}{6}(\frac{D}{2}+1)$ disjoint copies of an extremal P_m-free graph with maximum degree at most $D-1$.

By construction, every induced P_m in this graph contains a vertex in the central ring of cliques. Since that induced subgraph has no independent set of size q, the graph cannot have q independent m-vertex paths. When $m > 2$, $p(D-1, m)$ is at least $\binom{D}{2}$. In this case the edge contribution from the central ring of cliques has smaller order, and we ignore it.

When $m = 2$, there is a better construction. Build a ring of $3q-1$ independent sets, each of size $D/2$, each pair of adjoining sets inducing $K_{D/2,D/2}$. This graph is qP_2-free and has $(3q-1)D^2/4$ edges. □

These examples indicate that the essential step in determining the order of growth of $ex^*(D;H)$ as a function of D is determining the degree of the polynomial describing $p(D,m)$, for all m. Since the extremal values are known for $m \in \{2,3,4\}$ (and in [4] we will show that $ex^*(D;P_5) = \frac{2}{27}D^3 + O(D^2)$), the bounds above become

COROLLARY 3. The following lower and upper bounds hold for forbidden disjoint unions of $q \geq 2$ paths of the same length.

$$\frac{3q-1}{4}D^2 \leq ex^*(D;qP_2) \leq (2q-1)D^2 + O(D)$$
$$\frac{2q-1}{24}D^4 + O(D^3) \leq ex^*(D;qP_3) \leq \frac{3q-3}{2}D^4 + O(D^3)$$
$$\frac{2q-1}{12}D^4 + O(D^3) \leq ex^*(D;qP_4) \leq (4q-4)D^4 + O(D^3)$$
$$\frac{2q-1}{6\cdot 27}D^5 + O(D^4) \leq ex^*(D;qP_5) \leq \frac{2(5q-5)}{27}D^5 + O(D^4)$$

□

For $H = 2P_3$, the asymptotic value is $\frac{1}{8}D^4$, improving the upper bound of $(3/2)D^4$ above and attaining the lower bound. The extremal graph is a special case of the construction above. When $q = 2$, the three cliques in the central ring of cliques induce a single large clique. For comparison with other graphs that arise in the proof, it is helpful to consider a larger family of examples, of which this has the most edges.

Fixing D, let G_k be the connected graph consisting of a central clique K of order k and $k(D+1-k)$ peripheral cliques of order D, with each vertex of K adjacent to $D+1-k$ of the peripheral cliques by a single cut edge (see Fig. 1). The maximum degree of G_k is D, and G_k is $2P_3$-free, because every induced P_3 includes at least one vertex in K.

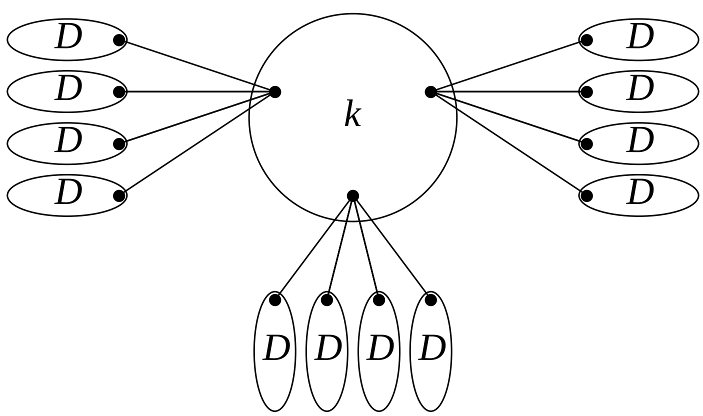

Fig. 1. The structure of the extremal graph

REMARK 4. For fixed D, the number of edges in G_k is maximized when $k = \lceil (D+1)/2 \rceil$, in which case the number of edges is

$$g(D) = \begin{cases} \frac{D^4+D^3+D^2+6D}{8} & \text{if } D \text{ is even} \\ \frac{D^4+D^3+2D^2+3D+1}{8} & \text{if } D \text{ is odd} \end{cases}$$

Proof: The number of edges in G_k is $k(D+1-k)[\binom{D}{2}+1]+\binom{k}{2} = \frac{1}{2}[k(D^3+D+1) - k^2(D^2-D+1)]$. This is a quadratic function in k with maximum at $k = \frac{(D^3+D+1)}{2(D^2-D+1)} = \frac{1}{2}(D+1+\frac{D}{D^2-D+1})$. Since k must be an integer, and a parabola is symmetric around its extreme, the maximum is achieved at the nearest integer to this, which is $\lceil (D+1)/2 \rceil$ if $D \geq 2$. When this value of k is used, we obtain the expression $g(D)$ above. □

3. STRUCTURAL DESCRIPTION

Let **G** denote the class of connected $2P_3$-free graphs with maximum degree D. Our approach to determining $ex^*(D; 2P_3)$ is to bound the size of an arbitrary $G \in \mathbf{G}$ in terms of $\omega(G)$ and compare this with values of $e(G_k)$ for appropriate k. In this section we develop a useful structural description for G.

If G is a clique, then $e(G) \leq \binom{D+1}{2} < g(D)$, so we may assume G is not a clique. Let W be a maximum clique of G, with order $w \geq 2$. Since G is connected, some vertex γ outside W has a neighbor β in W. Since W is a maximum clique, some vertex α in W is not adjacent to γ. These fixed vertices $P = \{\alpha, \beta, \gamma\}$ induce P_3.

Partition $V(G)$ into P, M_1, M_2, where $M_1 = N(P) - P$ and $M_2 = V(G) - N(P)$. Let H_i be the subgraph of G induced by M_i. Because there are no edges between M_2 and P, there can be no induced P_3 in H_2, so H_2 is a disjoint union of cliques. The components of H_2 may or may not be maximal cliques in G. We partition the set of components of H_2 into Q and Q', where Q contains those that are maximal cliques in G, and Q' contains those that are not maximal cliques in G (being contained in the neighborhood of some vertex of M_1). Given $x \in M_1$, let $H(x)$ be the set of components of H_2 that contain a neighbor of x, and let $Q(x) = H(x) \cap Q$ and $Q'(x) = H(x) \cap Q'$.

LEMMA 5. If $x, y \in M_1$ are nonadjacent, then $Q(x)$ and $Q(y)$ are ordered by inclusion.

Proof: If not, then there exist cliques $X \in Q(x)$ and $Y \in Q(y)$ such that X has no neighbor of y and Y has no neighbor of x. Now $\{x, X, y, Y\}$ together contain six vertices inducing $2P_3$. □

The next lemma will provide a bound on the size of Q.

LEMMA 6. The subgraph H_1 contains a clique K such that $Q = \cup_{x \in K} Q(x)$.

Proof: Among the cliques of H_1, let K be one that maximizes $|\cup_{x \in K} Q(x)|$, and let $S = \cup_{x \in K} Q(x)$. Suppose T is an element of Q omitted by S. Choose y such that $T \in Q(y)$. If $x \in K$ is a nonneighbor of y, then $Q(x)$ and $Q(y)$ are ordered by inclusion. With $T \in Q(y) - Q(x)$, we conclude $Q(x) \subseteq Q(y)$. Hence we can replace all the nonneighbors of y by y, and $K' = y \cup (K \cap N(y))$ contradicts the choice of K. □

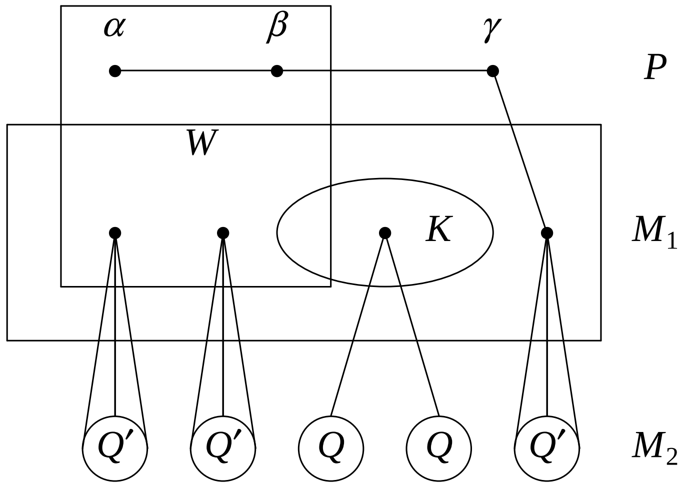

Fig. 2. The structure of a $2P_3$-free connected graph

This lemma is very helpful in bounding the size of Q. We will use a different method for Q', which requires another definition and lemma. We define a *webbed star* to be the join of a single ("central") vertex with a disjoint union of cliques. The central vertex is adjacent to all others.

LEMMA 7. Let H be a webbed star with $\Delta(H) \le d$ and $\omega(H) \le t$. Then $e(H) \le h(d,t)$, where

$$h(d,t) = \begin{cases} \frac{dt}{2} & \text{if } t \le \frac{d+1}{2} \\ \binom{t}{2} + \binom{d-t+2}{2} & \text{if } \frac{d+1}{2} \le t \le d+1 \\ \binom{d+1}{2} & \text{if } t \ge d+1 \end{cases}.$$

Proof: Let H be a webbed star, with central vertex v of degree d, and $\omega(H) = t$. Each vertex of $H - v$ has degree at most $t-1$. Hence $e(H) \le (d(t-1)+d)/2 = dt/2$.

We can improve this bound when $t > \frac{d+1}{2}$. Suppose X is a t-clique in H. Note that $v \in X$, and the vertices of $X - v$ have no other incident edges. There are $d-t+2$ vertices outside $X - v$, which is at most t if $t \ge d/2+1$. In this case, even with all possible edges present, we have $e(H) \le \binom{t}{2} + \binom{d-t+2}{2}$.

By simple algebra, $dt < t(t-1) + (d-t+2)(d-t+1)$ if and only if $t < d/2+1$. Furthermore, $h(d,t)$ is non-decreasing in d and t, so our computation for H also yields a bound for all webbed stars with smaller maximum degree or clique number. □

4. A SIMPLE EDGE BOUND

Our purported extremal configuration has cliques of order D. The relatively easy edge bound we obtain next will enable us to prove that an example with more edges must have clique number bigger than $\max\{D/2, D-6\}$. As we consider larger values of w, we must be more careful to avoid overcounting in order to bound the edge count by $g(D)$. For the case $w = D$, we will need a very careful argument. Throughout the remainder of the paper, we continue to use the notation developed in the previous section; in particular, clique K of order k contains an endpoint of an edge to each clique in Q, and W of order w (containing α, β) is a maximum clique.

LEMMA 8. If $G \in \mathbf{G}$, then $e(G) \leq (3D-2) + \binom{k}{2} + k(D-k)(1+\binom{w}{2}) + (3D-4-k)h(D-1,w)$.

Proof: To obtain the bound, we partition the edges into various sets. There are at most $3D-2$ edges incident to vertices of P and $\binom{k}{2}$ edges within K. Because each vertex of K has at least k neighbors in $P \cup M_1$, we conclude by Lemma 6 that $|Q| \leq k(D-k)$.

Consider the edges between K and M_2 plus the edges in $\cup_{x \in K} H(x)$; the bound on these is $k(D-k)(1+\binom{w}{2})$. There are at most $k(D-k)$ edges between K and M_2, and we have at most one clique in Q for each such edge. If any edge from K to M_2 is incident to a clique in Q' instead of Q, we are still counting a full $\binom{w}{2}$ for the clique in Q' that receives it, so we have also accounted for the edges of these cliques in Q'.

Each remaining edge is incident to at least one vertex of $M_1 - K$ or belongs to a clique in Q' contained in the neighborhood of some $x \in M_1 - K$. We consider the potential contribution for each such x. The edges incident to x, together with the edges of any cliques in Q' wholly contained in $N(x)$, form a webbed star. We have already accounted for an edge from x to P. Hence the number of edges in the webbed star of interest is bounded by $h(D-1,w)$. Since each vertex of $M_1 - K$ has a neighbor in P, and there are at most $3D-4$ edges from P to M_1, we have $|M_1 - K| \leq 3D-4$. We have shown that the specified terms account for all the edges. $\square$

LEMMA 9. If $G \in \mathbf{G}$ and $w \leq \max\{D/2, D-6\}$, then $e(G) < g(D)$.

Proof: The bound of Lemma 8 is increasing in w for each k; we show it is below $g(D)$ for the range of w specified here. The only term that can make a contribution quartic in D is $k(D-k)(1+\binom{w}{2})$, and the quartic contribution is at most $D^4/8$. Hence Lemma 8 already gives the asymptotic optimality of our construction. To show that a graph with small clique number cannot be larger, we must consider the cubic contribution from the last term.

If $w \leq D/2$, then $h(D-1,w) \leq D^2/4$ and $\binom{w}{2} \leq D(D-2)/8$. With these values, we maximize the function by setting $k = \lceil D/2 \rceil$. We obtain $e(G) \leq \frac{1}{32}(D^4 + 18D^3 - 20D^2 + 2D)$, which is less than $g(D)$ for $D \geq 4$. If $D/2 < w \leq D-5$, then $h(D-1,w) \leq \binom{w}{2} + \binom{6}{2}$. Again the bound is maximized at about $k/2$, where it is bounded by $\frac{1}{8}(D^4 - D^3 - 97D^2 + 800D)$. This is smaller than $g(D)$ whenever $D/2 < D-5$. $\square$

For the top few values of w, we must further reduce the cubic term in the bound by considering the role of the clique W. The next argument applies whenever $w \geq D/2+1$. We included the simple lemma above because the argument here for disjointness of W and K is not valid when $(D-w+2)/2 < w < D/2$.

LEMMA 10. If $G \in \mathbf{G}$ and $w \geq D/2+1$, then $e(G) \leq f(D,w,k)$, where

$$f(D,w,k) = 3D-2+\binom{w-2}{2}+\binom{k}{2}+k(D-k)(1+\binom{w}{2})$$

$$+(w-2)h(D-w+1,w)+(3D-2w-k)h(D-1,w).$$

Proof: We refine the argument of Lemma 8. Again there are at most $3D-2$ edges incident to vertices of P and $\binom{k}{2}$ within K. There are also $\binom{w-2}{2}$ edges within $W-\{\alpha,\beta\}$. Again the number of edges between K and M_2 plus those in $\cup_{x\in K}H(x)$ is at most $k(D-k)(1+\binom{w}{2})$. These contributions are the same as in Lemma 8, except that here we have also counted the edges within W. We claim that the last two terms in the formula bound the number of other edges.

Each edge that remains belongs to a webbed star centered at some vertex of M_1-K. The peripheral cliques of each such subgraph belong to Q' or are single vertices in M_1. Suppose first that $W\cap K=\emptyset$. Since the edges within W have already been counted, there remains degree at most $D-w+1$ for each vertex of W in the webbed star centered at it. The term $(w-2)h(D-w+1,w)$ bounds these contributions. There are at most $3D-2w-k$ vertices in M_1-K-W, each having at least one edge to P, so each has degree at most $D-1$ in the webbed star it contributes. The term $(3D-2w-k)h(D-1,w)$ bounds these contributions. Hence $f(D,w,k)$ is an upper bound on $e(G)$ when $W\cap K=\emptyset$.

Now suppose $|K\cap W|=r$; note that $r\leq w-2$. If $r>0$, we obtain a better upper bound than $f(D,w,k)$. For each vertex of $W-K$, we count $h(D-w+1,w)$, as before. For each vertex of M_1-K-W, we count $h(D-1,w)$, as before. For each vertex of $K-W$, we count $(D-k)(1+\binom{w}{2})$, as before. For each vertex of $K\cap W$, we have already counted the $(k-1)+(w-1)-(r-1)$ edges incident to it within K and W, so the degree that remains is at most $D-k-w+r+1$. Since these vertices are in K, we may have a full clique in Q for each such edge, so the contribution from each vertex of $K\cap W$ is at most $(D-k-w+r+1)(1+\binom{w}{2})$.

We have $|W-K|=w-r$ and $|K-W|=k-r$. Since vertices of $K\cap W$ already have neighbors in P by virtue of membership in W, we have r extra edges from P available, yielding $|M_1-K-W|\leq 3D-2w-k+r$. In comparison to $f(D,w,k)$, the contribution from M_1-K-W may be larger by $rh(D-1,w)$, and the contribution from $W-K$ is smaller by $rh(D-w+1,w)$. The r vertices of $K\cap W$ each can contribute only $D-k-w+r+1$ cliques of Q instead of the $D-k$ counted in $f(D,w,k)$, so here we lose $r(w-r-1)(1+\binom{w}{2})$.

Since $r\leq w-2$, we can show that the net change from $f(D,w,k)$ is negative when r is positive by proving that $h(D-1,w)-h(D-w+1,w)<1+\binom{w}{2}$. Because we may assume $D/2<w\leq D$, we need only consider one range for $h(D-w+1,w)$ and one range for $h(D-1,w)$. We have $h(D-1,w)-h(D-w+1,w)=\binom{w}{2}+\binom{D-w+1}{2}-\binom{D-w+2}{2}$ for w in this range. This is indeed less than $\binom{w}{2}$. □

LEMMA 11. If $G\in\mathbf{G}$, $e(G)\geq g(D)$ and $D\geq 5$, then $w=D$ and $W\cap K=\emptyset$.

Proof: By Lemma 9, we may assume $w \geq D/2+1$. In this range, the function $f(D,w,k)$ is $3D-2+\binom{w-2}{2}+\binom{k}{2}+k(D-k)(1+\binom{w}{2})+(w-2)\binom{D-w+2}{2}+(3D-2w-k)[\binom{w}{2}+\binom{D-w+1}{2}]$. The derivative of this function with respect to w is positive for all cases in this range. Hence for each choice of D,k, the function is maximized by maximizing w. When $w=D$, the function may exceed $g(D)$. When $w=D-1$, the function is maximized (when $D \geq 5$) by setting $k=\lfloor D/2 \rfloor$, where it equals $\frac{1}{8}[D^4-D^3+11D^2+2D-8]$ if D is even and $\frac{1}{8}[D^4-D^3+12D^2-3D-1]$ if D is odd. These quantities are less than $g(D)$.

Hence we may assume $w=D$. The value $f(D,D,k)$ is maximized by $k=\lfloor D/2 \rfloor$, where it equals $\frac{1}{8}[D^4+D^3+5D^2+10D-8]$ if D is even and $\frac{1}{8}[D^4+D^3+6D^2+7D-7]$ if D is odd. These values slightly exceed $g(D)$ (for other k, the values are less than $g(D)$). For $w=D$ and $k=\lfloor D/2 \rfloor$, we consider $r=|W\cap K|$. As computed in the preceding proof (after setting $w=D$), if $r>0$ we must reduce the edge bound by at least r times $(D-1-r)(1+\binom{D}{2})-(\binom{D}{2}-1)$. This is minimized by $r=1$, where it equals $\frac{1}{2}[D^3-4D^2+5D-2]$. When we subtract this from the largest value of $f(D,D,k)$, the bound is below $g(D)$ for $D \geq 4$. Hence we may assume $r=0$. □

5. THE DELICATE CASE

There remains the case $w=D$ and $W\cap K=\emptyset$. Our analysis is guided by the way in which the decomposition in Fig. 2 describes the graph G_{k+1} of Fig. 1. In G_{k+1}, the clique W must be a peripheral clique, and the central $(k+1)$-clique is $K\cup\gamma$. Hence it is natural that $f(D,D,k)$ is maximized at $k=\lfloor D/2 \rfloor$, since $e(G_k)$ is maximized at $k=\lceil (D+1)/2 \rceil$. In G_{k+1}, the vertex playing the role of γ has single edges to cliques; these cliques consist of one vertex of M_1 and one $(D-1)$-clique in Q'. In G_{k+1}, no components of H_2 contain neighbors of W, α has degree $D-1$, and no edges link K to the rest of M_1. The edges of G_{k+1} are completely counted by $3D-3+\binom{D-2}{2}+\binom{k}{2}+k(D-k)(1+\binom{D}{2})+(D-1-k)\binom{D}{2}$. This equals $e(G_{k+1})=(k+1)(D-k)(1+\binom{D}{2})+\binom{k+1}{2}$.

To obtain this bound, we must further refine our analysis of the restrictions on edges in graphs $G\in\mathbf{G}$ that have at least $g(D)$ edges. The amount by which $f(D,D,k)$ can exceed $g(D)$ is $\frac{1}{2}(D^2+D-2)$, whether D is even or odd. Hence it suffices to prove that $e(G)<f(D,D,k)-\frac{1}{2}(D^2+D-2)$ if $G\neq G_{k+1}$. We will do this by gradually forcing the structure to match G_{k+1}. We will show that the vertices of $W\cap M_1$ have no neighbors outside W, then that α

THEOREM 12. If $D\geq 8$, the maximum size of $G\in\mathbf{G}$ is $g(D)$, achieved uniquely by G_{k+1} for $k=\lfloor D/2 \rfloor$.

Proof: If $e(G)\geq g(D)$, we already have $w=D$ and $W\cap K=\emptyset$ in Fig. 2, by Lemma 11. We want to improve the bound $e(G)\leq f(D,D,k)$ by at least $\frac{1}{2}(D^2+D-2)=\binom{D}{2}+(D-1)$. The computation of $f(D,D,k)$ uses $|Q|\leq k(D-k)$. For each unit reduction from $k(D-k)$ in the edges from K yielding components in Q, we reduce the bound by $\binom{D}{2}$. Two such losses bring it below $g(D)$. Hence we may have $|H(x)|=D-1-k$ for one vertex x of K, but the remaining vertices of K must yield $D-k$ components in Q, and if there is such a "deficient" vertex the sets $H(x)$ are pairwise disjoint.

Numerically, comparison of $f(D,D,k)$ and $g(D)$ allows us to assume $3 \le k \le D-3$ if $D \ge 8$. With k in this range, the remarks just made about Q imply that $e(G) \ge g(D)$ requires the following: for every component of H_2, there is a vertex of K with no edge to that component, and every vertex of K has an edge to a clique in Q that has no edge from another vertex of K.

Let $W' = W \cap M_1$. Suppose some vertex $u \in W'$ has a neighbor v in M_2. Choose $z \in K$ such that z has no neighbor in the component of H_2 containing v. If there is only one choice for z, then $z \notin N(\alpha)$, else we have already reduced $k(D-k)$ by at least 2 (since $k \ge 3$). If there is more than one choice for z, we can pick one outside $N(\alpha)$, since α already has $D-1$ neighbors in W. Hence we can choose z and two vertices from an element of $Q(z)$, inducing P_3, all non-adjacent to the vertices α, u, v, which also induce P_3.

Hence we may assume that there is no edge between W' and M_2. If there is any edge between W' and $(M_1 - W) \cup \{\gamma\}$, then it is counted in the contributions to $f(D,D,k)$ for the vertex outside W'. Hence we may reduce the contribution to $f(D,D,k)$ from each webbed star centered at a vertex of W' from $h(D-w+1,w) = h(1,D) = 1$ to 0. This reduces the bound by $D-2$, after which a further reduction of $\binom{D}{2}+1$ will be sufficient to reach $g(D)$.

Let $M' = M_1 - K - W$. The contribution to $f(D,D,k)$ from each vertex of M' is at most $h(D-1,D) = \binom{D}{2}$. Since $f(D,D,k)$ allots $(3D-2w-k) = D-k$ for $|M'|$, any reduction in $|M'|$ reduces this contribution by $\binom{D}{2}$. If α, γ have a common neighbor in K, then $|M'| \le D-k-1$ and $|Q| < k(D-k)$, and we lose $\binom{D}{2}$ twice from the bound. If α, γ have a common neighbor $x \in M'$, then $|M'| \le D-k-1$. Also, the degree of x for its webbed star is now at most $D-2$, so this star contributes at most $\binom{D-1}{2}$ to the count instead of $\binom{D}{2}$. Together, we again have a sufficient reduction in the bound. Hence we may assume that α, γ have no common neighbor outside W.

At most one vertex in $M_1 - W$ is in $N(\alpha)$; the rest are in $N(\gamma)$. If α has a neighbor $x \in M'$ that has a neighbor $y \in K$, then again $|Q|$ goes down by one and x has degree at most $D-1$ in its webbed star, and the saving is sufficient. By the same computation, if $x \in K$ and $y \in M'$, then these vertices cannot be adjacent. Now it is possible to find an induced $2P_3$ using α, x in one path and γ, y in the other if α has a neighbor x outside W.

Hence we may assume that α has no neighbor outside W. This implies that $|M'| \le D-k-1$. Also, the number of edges incident to P is at most $3D-3$, not $3D-2$. This yields the additional desired reduction of $\binom{D}{2}+1$ in the bound. Now $e(G) \le g(D)$, and equality requires that all other contributions to $f(D,D,k)$ are achieved with equality. This prevents edges between W' and $M_1 - W'$, forces each vertex of the $D-1-k$ vertices of M' to belong to a pendant clique, and forces $k(D-k)$ cliques in Q. In short, it forces $G = G_{k+1}$. □

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