Independence Number of 2-Factor-Plus-Triangles Graphs

Douglas B. West

Department of Mathematics
University of Illinois at Urbana-Champaign
west@math.uiuc.edu
http://www.math.uiuc.edu/~west/pubs/publink.html

Joint work with
Jennifer Vandenbussche

Background

Def. A cycle-plus-triangles (CPT) graph is the union of a spanning cycle and a disjoint-triangles 2-factor.

Background

Def. A cycle-plus-triangles (CPT) graph is the union of a spanning cycle and a disjoint-triangles 2-factor.

Independence ratio of n-vertex G is $\alpha(G)/n$.

Conj. Du-Hsu-Hwang [1993] Every CPT graph has independence ratio $1/3$.

Background

Def. A cycle-plus-triangles (CPT) graph is the union of a spanning cycle and a disjoint-triangles 2-factor.

Independence ratio of n-vertex G is $\alpha(G)/n$.

Conj. Du-Hsu-Hwang [1993] Every CPT graph has independence ratio 1/3.

Conj. Erdös [1990] Every CPT graph is 3-colorable.

Background

Def. A cycle-plus-triangles (CPT) graph is the union of a spanning cycle and a disjoint-triangles 2-factor.

Independence ratio of n-vertex G is $\alpha(G)/n$.

Conj. Du-Hsu-Hwang [1993] Every CPT graph has independence ratio $1/3$.

Conj. Erdös [1990] Every CPT graph is 3-colorable.

Thm. Fleischner-Stiebitz [1992] Every CPT graph is 3-choosable. (Uses Alon-Tarsi Theorem.)

Background

Def. A cycle-plus-triangles (CPT) graph is the union of a spanning cycle and a disjoint-triangles 2-factor.

Independence ratio of n-vertex G is $\alpha(G)/n$.

Conj. Du-Hsu-Hwang [1993] Every CPT graph has independence ratio $1/3$.

Conj. Erdös [1990] Every CPT graph is 3-colorable.

Thm. Fleischner-Stiebitz [1992] Every CPT graph is 3-choosable. (Uses Alon-Tarsi Theorem.)

Thm. Sachs [1993] Every CPT graph is 3-colorable. (By combinatorial argument.)

A More General Family

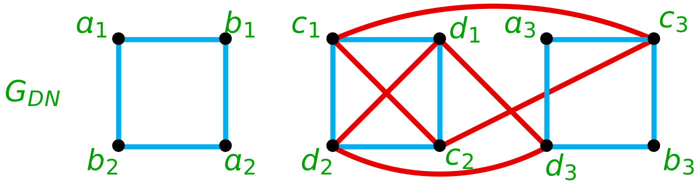
Def. A 2-factor-plus-triangles (2FPT) graph is the union of two 2-regular graphs on the same vertices, one consisting of triangles.

A More General Family

Def. A 2-factor-plus-triangles (2FPT) graph is the union of two 2-regular graphs on the same vertices, one consisting of triangles.

G_1 is 2-regular and G_2 has triples $T_x = \{x_1, x_2, x_3\}$.

Ex. Du-Ngo [2002] A 2FPT graph may contain K_4.

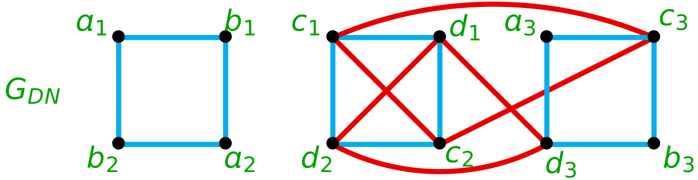


A More General Family

Def. A 2-factor-plus-triangles (2FPT) graph is the union of two 2-regular graphs on the same vertices, one consisting of triangles.

G_1 is 2-regular and G_2 has triples $T_x = \{x_1, x_2, x_3\}$.

Ex. Du-Ngo [2002] A 2FPT graph may contain K_4.



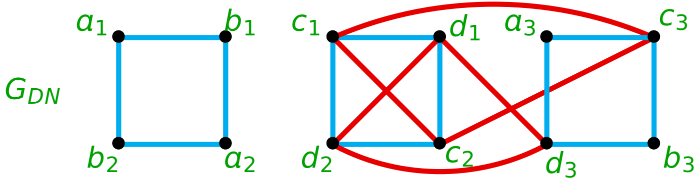
Is 2FPT graph 3-colorable if G_1 is C_4-free? (Erdös)

A More General Family

Def. A 2-factor-plus-triangles (2FPT) graph is the union of two 2-regular graphs on the same vertices, one consisting of triangles.

G_1 is 2-regular and G_2 has triples $T_x = \{x_1, x_2, x_3\}$.

Ex. Du-Ngo [2002] A 2FPT graph may contain K_4.



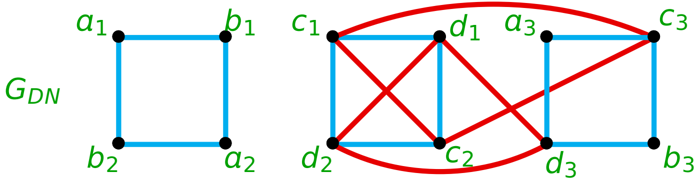
Is 2FPT graph 3-colorable if G_1 is C_4-free? (Erdös)
No! (Fleischner-Stiebitz; infinitely many 4-critical)

A More General Family

Def. A 2-factor-plus-triangles (2FPT) graph is the union of two 2-regular graphs on the same vertices, one consisting of triangles.

G_1 is 2-regular and G_2 has triples $T_x = \{x_1, x_2, x_3\}$.

Ex. Du-Ngo [2002] A 2FPT graph may contain K_4.



Is 2FPT graph 3-colorable if G_1 is C_4-free? (Erdös)
No! (Fleischner-Stiebitz; infinitely many 4-critical)

Note $\alpha(G_{DN}) = 3$, so indep. ratio can be as small as $1/4$.

Results for 2FPT Graphs

Since $\Delta(G) = 4$, Brooks' Theorem $\Rightarrow$ G is 4-colorable $\Rightarrow$ independence ratio $\geq 1/4$.

Results for 2FPT Graphs

Since $\Delta(G) = 4$, Brooks' Theorem $\Rightarrow$ G is 4-colorable
$\Rightarrow$ independence ratio $\geq 1/4$.

Thm. If a 2FPT graph has independence ratio $1/4$, then every component is the Du-Ngo graph G_{DN}.

Results for 2FPT Graphs

Since $\Delta(G) = 4$, Brooks' Theorem $\Rightarrow$ G is 4-colorable $\Rightarrow$ independence ratio $\geq 1/4$.

Thm. If a 2FPT graph has independence ratio $1/4$, then every component is the Du-Ngo graph G_{DN}.

Comment: Kierstead-Kostochka [2007] characterized G with $\Delta(G) = r$ s.t. every proper r-coloring is "equitable". For $r = 4$, there are three such connected graphs: G_{DN} and two of order 8 (not 2FPT graphs).

Results for 2FPT Graphs

Since $\Delta(G) = 4$, Brooks' Theorem $\Rightarrow$ G is 4-colorable $\Rightarrow$ independence ratio $\geq 1/4$.

Thm. If a 2FPT graph has independence ratio $1/4$, then every component is the Du-Ngo graph G_{DN}.

Thm. Infinitely many connected 2FPT graphs satisfy $\alpha(G)/n < 4/15$.

Results for 2FPT Graphs

Since $\Delta(G) = 4$, Brooks' Theorem $\Rightarrow$ G is 4-colorable $\Rightarrow$ independence ratio $\geq 1/4$.

Thm. If a 2FPT graph has independence ratio $1/4$, then every component is the Du-Ngo graph G_{DN}.

Thm. Infinitely many connected 2FPT graphs satisfy $\alpha(G)/n < 4/15$.

Conj. For $t < 4/15$, only finitely many connected 2FPT graphs satisfy $\alpha(G)/n \leq t$.

Results for 2FPT Graphs

Since $\Delta(G) = 4$, Brooks' Theorem $\Rightarrow$ G is 4-colorable $\Rightarrow$ independence ratio $\geq 1/4$.

Thm. If a 2FPT graph has independence ratio $1/4$, then every component is the Du-Ngo graph G_{DN}.

Thm. Infinitely many connected 2FPT graphs satisfy $\alpha(G)/n < 4/15$.

Conj. For $t < 4/15$, only finitely many connected 2FPT graphs satisfy $\alpha(G)/n \leq t$.

Thm. For 2FPT graphs, girth $7 \not\Rightarrow$ indep. ratio $1/3$.

Results for 2FPT Graphs

Since $\Delta(G) = 4$, Brooks' Theorem $\Rightarrow$ G is 4-colorable $\Rightarrow$ independence ratio $\geq 1/4$.

Thm. If a 2FPT graph has independence ratio $1/4$, then every component is the Du-Ngo graph G_{DN}.

Thm. Infinitely many connected 2FPT graphs satisfy $\alpha(G)/n < 4/15$.

Conj. For $t < 4/15$, only finitely many connected 2FPT graphs satisfy $\alpha(G)/n \leq t$.

Thm. For 2FPT graphs, girth $7 \not\Rightarrow$ indep. ratio $1/3$.

Conj. For 2FPT graphs, girth ≥ 8 $\Rightarrow$ indep. ratio $1/3$.

Characterizing 2FPT graphs with $\alpha(G)/n = 1/4$

Characterizing 2FPT graphs with $\alpha(G)/n = 1/4$

Lem. If $\exists$ indep. set I_0 with $3|I_0| > |N(I_0)|$, then $\frac{\alpha(G)}{n} > \frac{1}{4}$.

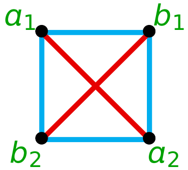
Pf. G connected $\Rightarrow$ can add x to I with nbr in $N(I)$.
Adding $|I|$ to $3|I| \geq |N(I)| + 1$ $\Rightarrow$ $|I| \geq (n+1)/4$.

Characterizing 2FPT graphs with $\alpha(G)/n = 1/4$

Lem. If $\exists$ indep. set I_0 with $3|I_0| > |N(I_0)|$, then $\frac{\alpha(G)}{n} > \frac{1}{4}$.

Pf. G connected $\Rightarrow$ can add x to I with nbr in $N(I)$.
Adding $|I|$ to $3|I| \geq |N(I)| + 1 \Rightarrow |I| \geq (n+1)/4$.

Lem. A copy of K_4 in G arises only as a 4-cycle in G_1 plus chords from distinct triples.

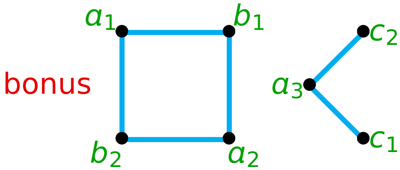


Characterizing 2FPT graphs with $\alpha(G)/n = 1/4$

Lem. If $\exists$ indep. set I_0 with $3|I_0| > |N(I_0)|$, then $\frac{\alpha(G)}{n} > \frac{1}{4}$.

Pf. G connected $\Rightarrow$ can add x to I with nbr in $N(I)$.
Adding $|I|$ to $3|I| \geq |N(I)| + 1 \Rightarrow |I| \geq (n+1)/4$.

Lem. A copy of K_4 in G arises only as a 4-cycle in G_1 plus chords from distinct triples.



Lem. If G has a bonus 4-clique, then $\alpha(G) > n/4$.

Pf. Set $I_0 = \{b_1, a_3, c_3\}$.

2FPT graphs with $\alpha(G) = n/4$, cont.

Lem. If G is 4-reg. with no 4-clique, then $\exists$ indep. set I with $3|I| > |N(I)|$ or with $3|I| = N(I)$ and $|I| < n/4$.

Pf. (Idea) Find even cycle C with at most one chord. If $|V(C)| \neq n/2$, then one indep. set using half of $V(C)$ has the desired property. $|V(C)| = n/2$ is a special case.

2FPT graphs with $\alpha(G) = n/4$, cont.

Lem. If G is 4-reg. with no 4-clique, then $\exists$ indep. set I with $3|I| > |N(I)|$ or with $3|I| = N(I)$ and $|I| < n/4$.

Pf. (Idea) Find even cycle C with at most one chord. If $|V(C)| \neq n/2$, then one indep. set using half of $V(C)$ has the desired property. $|V(C)| = n/2$ is a special case.

- When G has 4-cliques, algorithm produces I such that
1) $I \cup N(I)$ contains all 4-cliques and
2a) $3|I| > |N(I)|$ or 2b) $3|I| = N(I)$ and $|I| < n/4$.

2FPT graphs with $\alpha(G) = n/4$, cont.

Lem. If G is 4-reg. with no 4-clique, then $\exists$ indep. set I with $3|I| > |N(I)|$ or with $3|I| = N(I)$ and $|I| < n/4$.

Pf. (Idea) Find even cycle C with at most one chord. If $|V(C)| \neq n/2$, then one indep. set using half of $V(C)$ has the desired property. $|V(C)| = n/2$ is a special case.

- When G has 4-cliques, algorithm produces I such that

1) $I \cup N(I)$ contains all 4-cliques and
2a) $3|I| > |N(I)|$ or 2b) $3|I| = N(I)$ and $|I| < n/4$.

Thm. $\alpha(G) > n/4$.

Pf. Continue giving I a vertex $x \in I \cup N(I)$ adding least to $N(I)$. Claim: Last step adds ≤ 2 to $N(I)$, so $3|I| > |N(I)|$.

2FPT graphs with $\alpha(G) = n/4$, cont.

Lem. If G is 4-reg. with no 4-clique, then $\exists$ indep. set I with $3|I| > |N(I)|$ or with $3|I| = N(I)$ and $|I| < n/4$.

Pf. (Idea) Find even cycle C with at most one chord. If $|V(C)| \neq n/2$, then one indep. set using half of $V(C)$ has the desired property. $|V(C)| = n/2$ is a special case.

- When G has 4-cliques, algorithm produces I such that

1) $I \cup N(I)$ contains all 4-cliques and
2a) $3|I| > |N(I)|$ or 2b) $3|I| = N(I)$ and $|I| < n/4$.

Thm. $\alpha(G) > n/4$.

Pf. Continue giving I a vertex $x \in I \cup N(I)$ adding least to $N(I)$. Claim: Last step adds ≤ 2 to $N(I)$, so $3|I| > |N(I)|$.

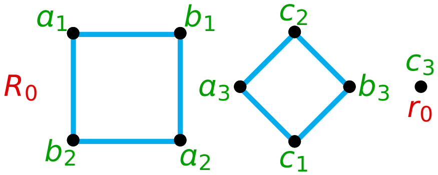
If not, and last added x adds u, v, w to $N(I)$, then u, v, or w also adds 3 to $N(I)$, so $\{u, v, w, x\}$ is a 4-clique.

Connected 2FPT Graphs with $\alpha(G)/n < 4/15$

Construction: First build R_i with root r_i for $i \geq 0$.

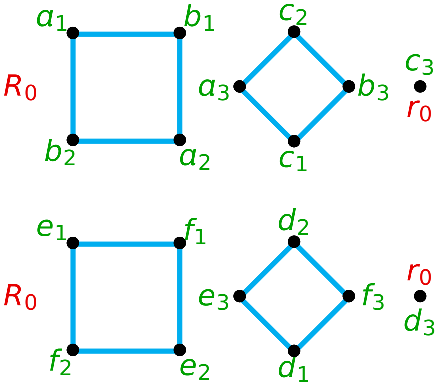
Connected 2FPT Graphs with $\alpha(G)/n < 4/15$

Construction: First build R_i with root r_i for $i \geq 0$.



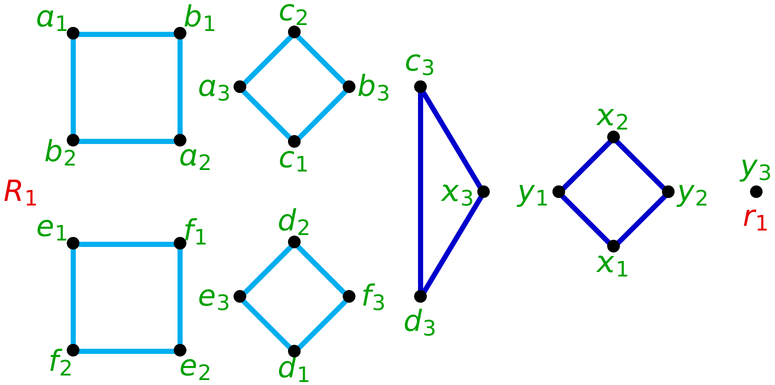
Connected 2FPT Graphs with $\alpha(G)/n < 4/15$

Construction: First build R_i with root r_i for $i \geq 0$.



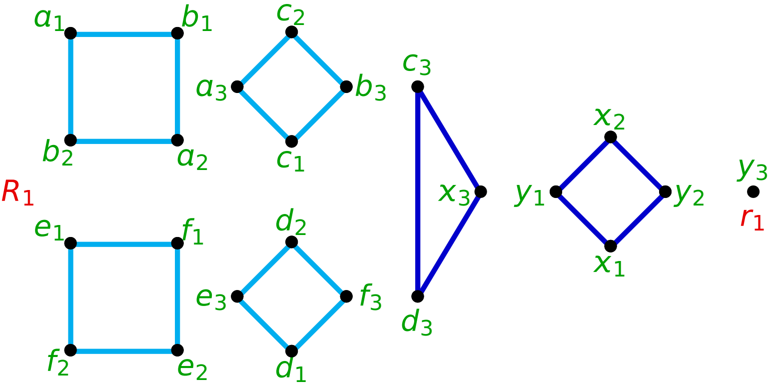
Connected 2FPT Graphs with $\alpha(G)/n < 4/15$

Construction: First build R_i with root r_i for $i \geq 0$.



Connected 2FPT Graphs with $\alpha(G)/n < 4/15$

Construction: First build R_i with root r_i for $i \geq 0$.



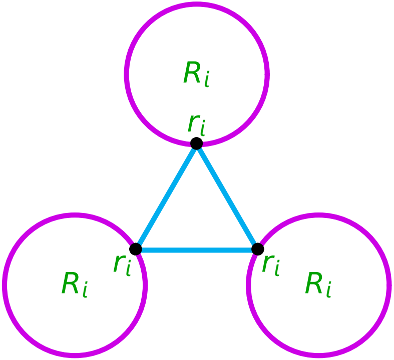
$n_i = 15(2^i) - 6$. $(n_0 = 9)$

R_i is 2-factor on $R_i - r_i$ plus $n_i/3$ disjoint triangles.

$\alpha(R_i - r_i) = 4(2^i) - 2$, using no nbrs of r_i. $(\alpha_0 = 2)$.

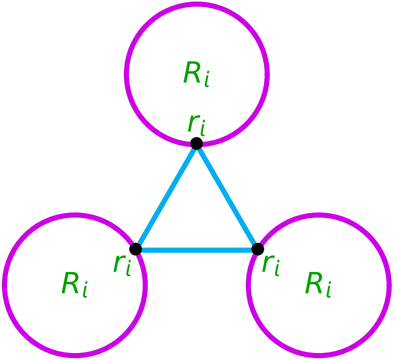
The graph Q_i

$$|V(Q_i)| = 3|V(R_i)| \quad \text{and} \quad \alpha(Q_i) = 3\alpha(R_i) + 1$$



The graph Q_i

$$|V(Q_i)| = 3|V(R_i)| \quad \text{and} \quad \alpha(Q_i) = 3\alpha(R_i) + 1$$



Thm. 2FPT graph Q_i has indep. ratio $\frac{4(2^i)-5/3}{15(2^i)-6}$.

Larger Girth

Def. (Alon [1998], Fellows [1990]) A graph G is strongly r-colorable if for every partition of $V(G)$ into sets of size r, G has a proper r-coloring with no color appearing twice in one set of the partition.

Larger Girth

Def. (Alon [1998], Fellows [1990]) A graph G is strongly r-colorable if for every partition of $V(G)$ into sets of size r, G has a proper r-coloring with no color appearing twice in one set of the partition.

Erdös's question: Are triangle-free 2-regular graphs strongly 3-colorable?

Larger Girth

Def. (Alon [1998], Fellows [1990]) A graph G is strongly r-colorable if for every partition of $V(G)$ into sets of size r, G has a proper r-coloring with no color appearing twice in one set of the partition.

Erdös's question: Are triangle-free 2-regular graphs strongly 3-colorable?

Thm. 2-regular graphs with girth 7 need not be strongly 3-colorable. They may have indep. ratio $< \frac{1}{3}$.

Larger Girth

Def. (Alon [1998], Fellows [1990]) A graph G is strongly r-colorable if for every partition of $V(G)$ into sets of size r, G has a proper r-coloring with no color appearing twice in one set of the partition.

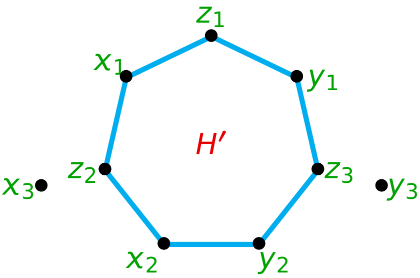
Erdös's question: Are triangle-free 2-regular graphs strongly 3-colorable?

Thm. 2-regular graphs with girth 7 need not be strongly 3-colorable. They may have indep. ratio $< \frac{1}{3}$.

Conj. 2-regular graphs with girth ≥ 8 are strongly 3-colorable.

Girth 7 and indep. ratio < 1/3

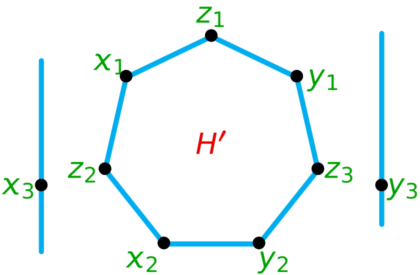
Thm. connected 2FPT graph w. girth 7 and $\frac{\alpha(G)}{n} = \frac{20}{63}$.



An indep. set has ≤ 2 vertices from the 7-cycle.

Girth 7 and indep. ratio < 1/3

Thm. connected 2FPT graph w. girth 7 and $\frac{\alpha(G)}{n} = \frac{20}{63}$.

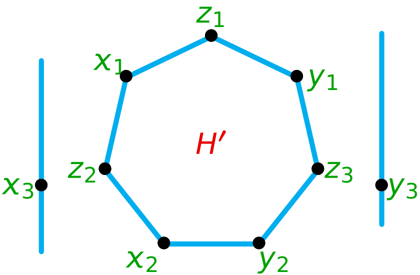


An indep. set has ≤ 2 vertices from the 7-cycle.

Take $7k$ copies of H'. For $0 \leq j \leq k-1$, add two cycles:
on copies $7j+1, \ldots, 7j+7 \pmod{7k}$ of x_3,
on copies $7j, \ldots, 7j+6 \pmod{7k}$ of y_3.

Girth 7 and indep. ratio < 1/3

Thm. connected 2FPT graph w. girth 7 and $\frac{\alpha(G)}{n} = \frac{20}{63}$.



An indep. set has ≤ 2 vertices from the 7-cycle.

Take $7k$ copies of H'. For $0 \leq j \leq k-1$, add two cycles:
on copies $7j+1, \ldots, 7j+7 \pmod{7k}$ of x_3,
on copies $7j, \ldots, 7j+6 \pmod{7k}$ of y_3.

Each added 7-cycle contributes at most 3 vertices.
$\therefore$ each group of 7 copies of H' contributes at most 20.

Triangles-Plus-Triangles Graphs are 3-choosable

Thm. If G_1 and G_2 are graphs whose components have at most s vertices, then $G_1 \cup G_2$ is s-choosable.

Triangles-Plus-Triangles Graphs are 3-choosable

Thm. If G_1 and G_2 are graphs whose components have at most s vertices, then $G_1 \cup G_2$ is s-choosable.

Pf. Let $G = G_1 \cup G_2$ (may assume $V(G_1) = V(G_2) = G$). Each $v \in V(G)$ has a list $L(v)$ of s available colors.

Triangles-Plus-Triangles Graphs are 3-choosable

Thm. If G_1 and G_2 are graphs whose components have at most s vertices, then $G_1 \cup G_2$ is s-choosable.

Pf. Let $G = G_1 \cup G_2$ (may assume $V(G_1) = V(G_2) = G$). Each $v \in V(G)$ has a list $L(v)$ of s available colors.

Let H be the intersection graph of the components in G_1 and G_2; a bipartite multigraph with maxdegree $\leq s$.

Triangles-Plus-Triangles Graphs are 3-choosable

Thm. If G_1 and G_2 are graphs whose components have at most s vertices, then $G_1 \cup G_2$ is s-choosable.

Pf. Let $G = G_1 \cup G_2$ (may assume $V(G_1) = V(G_2) = G$). Each $v \in V(G)$ has a list $L(v)$ of s available colors.

Let H be the intersection graph of the components in G_1 and G_2; a bipartite multigraph with maxdegree $\leq s$.

H has one edge for each $v \in V(G)$; give it list $L(v)$.

Triangles-Plus-Triangles Graphs are 3-choosable

Thm. If G_1 and G_2 are graphs whose components have at most s vertices, then $G_1 \cup G_2$ is s-choosable.

Pf. Let $G = G_1 \cup G_2$ (may assume $V(G_1) = V(G_2) = G$). Each $v \in V(G)$ has a list $L(v)$ of s available colors.

Let H be the intersection graph of the components in G_1 and G_2; a bipartite multigraph with maxdegree $\leq s$.

H has one edge for each $v \in V(G)$; give it list $L(v)$.

By Galvin's Theorem, H has a proper edge-coloring with the color of each edge chosen from its list. This gives vertices in the same component in G_1 (or G_2) distinct colors. Now G is properly colored from the lists. ∎