# Independence number of 2-factor-plus-triangles graphs

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**Abstract**

A *2-factor-plus-triangles graph* is the union of two 2-regular graphs $G_1$ and $G_2$ with the same vertices, such that $G_2$ consists of disjoint triangles. Let $\mathcal{G}$ be the family of such graphs. These include the famous "cycle-plus-triangles" graphs shown to be 3-choosable by Fleischner and Stiebitz. The independence ratio of a graph in $\mathcal{G}$ may be less than 1/3; but achieving the minimum value 1/4 requires each component to be isomorphic to a single 12-vertex graph. Nevertheless, $\mathcal{G}$ contains infinitely many connected graphs with independence ratio less than 4/15. For each odd $g$ there are infinitely many connected graphs in $\mathcal{G}$ such that $G_1$ has girth $g$ and the independence ratio of $G$ is less than 1/3. Also, when 12 divides $n$ (and $n \neq 12$) there is an $n$-vertex graph in $\mathcal{G}$ such that $G_1$ has girth $n/2$ and $G$ is not 3-colorable. Finally, unions of two graphs whose components have at most $s$ vertices are $s$-choosable.

## 1 Introduction

The Cycle-Plus-Triangles Theorem of Fleischner and Stiebitz [5] states that if a graph $G$ is the union of a spanning cycle and a 2-factor consisting of disjoint triangles, then $G$ is 3-choosable, where a graph is *k-choosable* if for every assignment of lists of size $k$ to the vertices, there is a proper coloring such that the color on each vertex is chosen from its list. Sachs [8] gave a proof by elementary methods that all such graphs are 3-colorable. Both results imply an earlier conjecture by Du, Hsu, and Hwang [1], stating that a cycle-plus-triangles graph with $3k$ vertices has independence number $k$. Erdős [3] strengthened the conjecture to the more well-known statement that these graphs are 3-colorable. We return to the original topic of independence number but study it on a more general family of graphs.

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A *2-factor-plus-triangles graph* is a union of two 2-regular graphs $G_1$ and $G_2$ on the same vertex set, where the components of $G_2$ are triangles. Note that $G_1$ and $G_2$ may share edges. For such a graph $G$, we denote the vertex sets of the components of $G_2$ as $T_1, \ldots, T_k$, with $T_x = \{x_1, x_2, x_3\}$, and we refer to $T_x$ as a "triple" to distinguish it from a 3-cycle in $G_1$. When $G_1$ is a single cycle, $G$ is a cycle-plus-triangles graph.

Let $\mathcal{G}$ denote the family of 2-factor-plus-triangles graphs. It is easy to construct graphs in $\mathcal{G}$ that contain $K_4$ (see Figure 1, for example), so graphs in $\mathcal{G}$ need not be 3-colorable. Erdos [3] asked if a graph in $\mathcal{G}$ is 3-colorable whenever its factor $G_1$ is $C_4$-free. Fleischner and Stiebitz [6] answered this negatively, citing an infinite family of such graphs in $\mathcal{G}$ that are 4-critical, due to Gallai. In fact, graphs in $\mathcal{G}$ with $3k$ vertices may fail to have an independent set of size $k$, such as the graph in Figure 1 due to Du and Ngo [2]. Here we draw only $G_1$ and indicate the triples $T_a, T_b, T_c, T_d$ using subscripted indices.

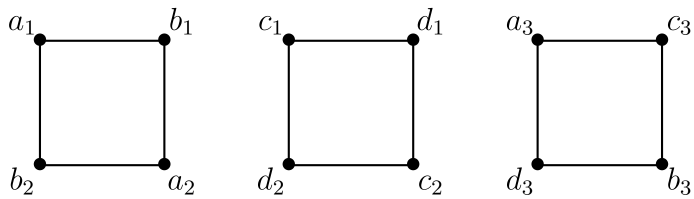


Figure 1: The Du-Ngo graph $G_{DN}$, omitting triangles on sets of the form $\{x_1, x_2, x_3\}$.

An *independent set* is a set of pairwise nonadjacent vertices. The *independence number* $\alpha(G)$ of a graph $G$ is the maximum size of such a set in $G$.

**Proposition 1.1.** *The independence number of the Du-Ngo graph $G_{DN}$ is* 3.

*Proof.* An independent set $S$ in $G_{DN}$ contains at most one vertex from the sets $\{a_1, b_1, a_2, b_2\}$ and $\{c_1, d_1, c_2, d_2\}$, since each is a 4-clique. Further, $S$ contains two vertices of $(a_3, b_3, c_3, d_3)$ only if it avoids one of the 4-cliques. Thus $|S| \leq 3$, and $\{a_1, c_1, d_3\}$ achieves the bound. $\square$

The *independence ratio* of an $n$-vertex graph $G$ is $\alpha(G)/n$. Proposition 1.1 states that the independence ratio of $G_{DN}$ is $1/4$. Because graphs in $\mathcal{G}$ have maximum degree at most 4 and do not contain $K_5$, Brooks' Theorem implies that every graph in $\mathcal{G}$ has independence ratio at least $1/4$. We characterize those graphs achieving equality in this easy bound; they are the graphs in which every component is $G_{DN}$. We produce larger independent sets for all other graphs in $\mathcal{G}$. We also construct infinitely many connected graphs in $\mathcal{G}$ with independence ratio less than $4/15$. However, we conjecture that for any $t$ less than $4/15$, only finitely many connected graphs in $\mathcal{G}$ have independence ratio at most $t$.

In light of Erdős' question about 3-colorability of graphs in $\mathcal{G}$ when $G_1$ has no 4-cycle, we study the independence ratio under girth restrictions for $G_1$. For any odd $g$, we construct infinitely many connected examples in which the girth of $G_1$ is $g$ and yet the independence ratio is less than 1/3; it can be as small as $\frac{1}{3} - \frac{1}{g^2+2g}$ when $g \equiv 1 \mod 6$. The number of vertices in each example is more than $g^2$, and we conjecture that the independence ratio of $G$ is 1/3 when $G_1$ has girth at least $\sqrt{|V(G)|}$. On the other hand, no girth threshold less than $|V(G)|$ can guarantee 3-colorability; when the number of vertices is a nontrivial multiple of 12, we construct examples where $G_1$ consists of just two cycles of equal length but $G$ is not 3-colorable.

Finally, we show that if $G$ is a union of two graphs whose components have at most $s$ vertices, then $G$ is $s$-choosable; this yields 3-choosability for graphs in $\mathcal{G}$ where the components of $G_1$ are all 3-cycles. This last result is an easy consequence of the $s$-choosability of the line graphs of bipartite graphs.

Our graphs have no multiple edges; when $G_1$ and $G_2$ share an edge, its vertices have degree less than 4 in the union. For a graph $G$ and a vertex $x \in V(G)$, the *neighborhood* $N_G(x)$ is the set of vertices adjacent to $x$ in $G$, and a *$G$-neighbor* of $x$ is an element of $N_G(x)$. For $S \subseteq V(G)$, we let $N_G(S) = \bigcup_{x\in S} N_G(x)$. If $A$ and $B$ are sets, then $A-B = \{a \in A\colon\ a \notin B\}$.

## 2 Independence ratio at least 1/4

The independence number of a graph is the sum of the independence numbers of its components. Therefore, to characterize the graphs in $\mathcal{G}$ with independence ratio 1/4, it suffices prove that every connected graph in $\mathcal{G}$ other than $G_{DN}$ has independence ratio larger than 1/4. Let $\mathcal{G}' = \{G \in (\mathcal{G} - \{G_{DN}\})\colon\ G \text{ is connected}\}$.

Proving this is surprisingly difficult. We present an algorithm to produce a sufficiently large independent set for any $G \in \mathcal{G}'$. A simple greedy algorithm finds an independent set with almost 1/4 of the vertices; it will be applied to prove the full result. This simple algorithm maintains an independent set $I$ and the set $S$ of neighbors of $I$.

**Algorithm 2.1.** Given an initial independent set $I$ in $G$, let $S = N_G(I)$. While $I \cup S \neq V(G)$, choose $v \in V(G) - (I \cup S)$ to minimize $|N(v) - S|$, and add $v$ to $I$ and $N_G(v)$ to $S$.

**Lemma 2.2.** *If $G$ is an $n$-vertex graph in $\mathcal{G}'$, then $\alpha(G) \geq (n-1)/4$. If $G$ has an independent set $I_0$ with $3|I_0| > |N_G(I_0)|$, then $\alpha(G) > n/4$.*

*Proof.* Initialize Algorithm 2.1 with $I$ as any single vertex in $G$; this puts at most 4 vertices in $S$. At each subsequent step, some vertex $v$ outside $I \cup S$ has a neighbor in $S$, since $G$ is connected and $N_G(I) = S$. Hence each step adds at most 3 vertices to $S$ and 1 vertex to $I$. Therefore, $|S| \leq 3|I| + 1$ when the algorithm ends. Since $n = |I| + |S|$ at that point, we conclude that $|I| \geq (n-1)/4$.

If $3|I_0| > |N_G(I_0)|$, then initializing Algorithm 2.1 with $I = I_0$ (and $S = N_G(I_0)$) yields $|S| \leq 3|I| - 1$ at the end by the same computation, and hence $|I| \geq (n+1)/4$. □

In order to push the independence ratio above 1/4, we will preface Algorithm 2.1 with another algorithm that will choose the initial independent set more carefully, seeking an independent set $I_0$ as in Lemma 2.2 or one that will lead to a gain later under Algorithm 2.1.

First we characterize how 4-cliques can arise in graphs in $\mathcal{G}$ (a *k-clique* is a set of $k$ pairwise adjacent vertices).

**Lemma 2.3.** *A 4-clique in a graph $G$ in $\mathcal{G}$ arises only as the union of a 4-cycle in $G_1$ and disjoint edges from two triples in $G_2$ (see Figure 2).*

*Proof.* Let $X$ be a 4-clique in $G$. Since $G_1$ contributes at most two edges to each vertex, each vertex in $X$ has a $G_2$-neighbor in $X$. In particular, no triple in $G_2$ is contained in $X$, and $X$ must have the form $\{a_1, a_2, b_1, b_2\}$ for some $T_a$ and $T_b$. To make $X$ pairwise adjacent, $a_1, b_1, a_2, b_2$ in order must form a 4-cycle in $G_1$. □

We define a substructure that yields a good independent set for the initialization of Algorithm 2.1. A *bonus 4-clique* in a graph in $\mathcal{G}$ is a 4-clique $Q$ such that for some triple $T_a$ contributing two vertices to $Q$, the vertices of $N_{G_1}(a_3)$ lie in the same triple. Figure 2 illustrates the definition.

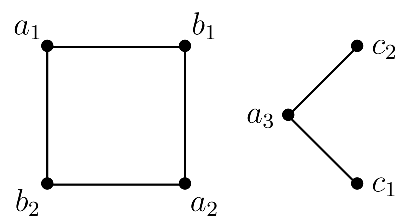


Figure 2: A bonus 4-clique

**Lemma 2.4.** *Let $G$ be an $n$-vertex graph in $\mathcal{G}'$. If $G$ has a bonus 4-clique, then $\alpha(G) > n/4$.*

*Proof.* Consider a bonus 4-clique, labeled as in Figure 2 without loss of generality. The set $\{b_1, a_3, c_3\}$ is independent, and its neighborhood is $\{a_1, a_2, b_2, b_3, c_1, c_2\} \cup N_{G_1}(c_3)$. Thus setting $I_0 = \{b_1, a_3, c_3\}$ in Lemma 2.2 yields the conclusion. □

A *block* of a graph is a maximal subgraph that contains no cut-vertex. Two blocks in a graph share at most one vertex, and a vertex in more than one block is a cut-vertex. A *leaf block* of a graph $G$ is a block that has at most one vertex shared with other blocks of $G$. We need a structural result to extract large independent sets from leaf blocks.

**Lemma 2.5.** *Let $G$ be an $n$-vertex 4-regular graph in $\mathcal{G}'$. If $G$ has no 4-clique, then $G$ has an independent set $I$ such that $3|I| > |N_G(I)|$ or such that $3|I| = |N_G(I)|$ and $|I| < n/4$.*

*Proof.* Every vertex of $G$ lies in a triple, and every triple lies in a block of $G$. Since $G$ is 4-regular, a leaf block contains a triple and at least one more vertex. A shortest path joining two vertices of the triple that uses a vertex outside the triple yields an even cycle with at most one chord. (Note: Erdős, Rubin, and Taylor [4] showed by a harder proof that all 2-connected graphs other than complete graphs and odd cycles have such a cycle.)

An independent set $I$ with $|I| > n/4$ vertices satisfies $3|I| > |N_G(I)|$ and hence suffices.

We may assume that $G$ has no 4-cycle, since $G$ has no 4-clique and a 4-cycle in $G$ with at most one chord has an independent set $I$ with $3|I| = |N_G(I)|$ and $|I| = 2 \neq n/4$ (note that $3 \mid n$). If $C$ is an even cycle in $G$ having at most one chord, then at least one of the two maximum independent sets in $C$ contains at most one vertex of such a chord and is independent in $G$. Let $I$ be such an independent set.

Since each vertex of $I$ has at least two neighbors on $C$ and at most two outside it, $3|I| \geq |N_G(I)|$. We have found the desired set $I$ unless $|I| = n/4$. In this case, let $T = V(G) - V(C)$. If $I$ is not a maximal independent set, then $\alpha(G) > n/4$, so we may assume that every vertex of $T$ has a neighbor in $I$. Since $I \subseteq V(C)$, each vertex in $I$ has at most two neighbors in $T$. Hence each vertex of $T$ has exactly one neighbor in $I$, and each vertex of $I$ has two neighbors in $T$ (and $C$ has no chord).

Let $u, v, w$ be three consecutive vertices along $C$, with $u, w \in I$. Let $\{x, x'\} = N_G(u) \cap T$ and $\{y, y'\} = N_G(w) \cap T$. If $xx' \notin E(G)$, then replacing $u$ with $\{x, x'\}$ in $I$ yields $\alpha(G) > n/4$. Hence we may assume that $xx' \in E(G)$, and similarly $yy' \in E(G)$. If $v$ has a neighbor in $\{x, x', y, y'\}$, then $G$ has a 4-cycle, which we excluded. Since $G$ has no 4-clique, some vertex in $\{x, x'\}$ has a nonneighbor in $\{y, y'\}$, say $xy \notin E(G)$. Now replacing $\{u, w\}$ with $\{v, x, y\}$ in $I$ yields $\alpha(G) > n/4$. □

We now present an algorithm to apply before Algorithm 2.1, as "preprocessing". The proof of Lemma 2.5 can be implemented as an algorithm used by Algorithm 2.6 for the case where $G$ has no 4-clique. Like Algorithm 2.1, Algorithm 2.6 maintains an independent set $I \subseteq V(G)$ and the set $S$ of its neighbors. It produces a nonempty independent set $I$ such that $3|I| > |S|$ or such that $3|I| = |S| < 3n/4$ and all vertices of 4-cliques lie in $I \cup S$.

After Algorithm 2.6, we apply Algorithm 2.1 starting with this set as $I$. Lemma 2.2 implies that if $3|I| > |S|$, then $\alpha(G) > n/4$. We will show in Theorem 2.8 that if $3|I| = |S|$, then the exhaustion of the 4-cliques during Algorithm 2.6 will guarantee the existence of a step in Algorithm 2.1 in which $S$ gains at most two vertices. Thus again we will have $3|I| > |S|$ and $|I| > n/4$ at the end.

To facilitate the description of Algorithm 2.6, we introduce several definitions. A triple having two vertices in a 4-clique is a *clique-triple*. Two clique-triples that contribute two vertices each to the same 4-clique (see Lemma 2.3) are *mates*. If $T_a$ intersects a 4-clique $Q$, but $I \cup S$ does not intersect $T_a \cup Q$, then $T_a$ is a *free clique-triple*.

**Algorithm 2.6.** Given an $n$-vertex graph $G$ in $\mathcal{G}'$, initialize $I = S = \varnothing$. Maintain $S = N_G(I)$. When we "stop", the current set $I$ is the output.

Suppose first that $G$ has no 4-clique. If $E(G_1) \cap E(G_2) \neq \varnothing$, then let $I$ consist of one endpoint of such an edge and stop. Otherwise, $G$ is 4-regular; let $I$ be an independent set produced by the algorithmic implementation of Lemma 2.5, and stop.

If $G$ has a bonus 4-clique, then define $I$ as in Lemma 2.4 and stop.

If $G$ has a 4-clique but no bonus 4-clique, then repeat the steps below until either $3|I| > |S|$ or $I \cup S$ contains all vertices of 4-cliques; then stop.

1. If some vertex outside $I \cup S$ has at most two neighbors outside $S$, add it to $I$ and stop.

2. If there is a free clique-triple $T_a$ with mate $T_b$ such that $S$ contains $b_3$ or some $G_1$-neighbor of $a_3$, then add $\{a_3, b_1\}$ to $I$ and stop.

3. Otherwise, let $T_a$ be a free clique-triple with mate $T_b$, and let $N_{G_1}(a_3) = \{c_3, d_3\}$. Since $G$ has no bonus 4-clique, $c \neq d$. If $\{c_1, d_1, c_2, d_2\}$ is not a 4-clique in $G$, then add $\{a_3, b_1\}$ to $I$. If $\{c_1, d_1, c_2, d_2\}$ is a 4-clique in $G$, then add $\{a_3, b_1, c_3, d_1\}$ to $I$.

**Lemma 2.7.** *For $G \in \mathcal{G}'$, Algorithm 2.6 produces an independent set $I$ with neighborhood $S$ such that $3|I| > |S|$ or such that $3|I| = |S|$ and $I \cup S$ contains all 4-cliques in $G$.*

*Proof.* First suppose that $G$ has no 4-clique. If $G$ is 4-regular, then Algorithm 2.6 uses the construction of Lemma 2.5 to produce $I$ such that $3|I| > |S|$ or such that $3|I| = |S|$ and $|I| < n/4$ (and hence $I \cup S \neq V(G)$). If $G$ is not 4-regular, then it finds such a set of size 1.

If $G$ has a bonus 4-clique, then the independent set $I$ is as in the proof of Lemma 2.4, with $3|I| > |S|$.

Therefore, we may assume that $G$ has a 4-clique but no bonus 4-clique. In this case, the algorithm iterates Step 3 until it reaches a state where Step 1 or 2 applies or it runs out of free clique-triples.

To show that ending in Step 1 or 2 yields the desired conclusion, suppose that each instance of Step 3 maintains $3|I| \geq |S|$. In Step 1, we then add one vertex to $I$ and at most two to $S$. In Step 2, we add $\{a_3, b_1\}$ to $I$ and $\{a_1, a_2, b_2, b_3\} \cup N_{G_1}(a_3)$ to $S$, but $S$ already contains at least one of these six vertices.

Hence we must show that Step 3 maintains $3|I| \geq |S|$. To avoid getting stuck by running out of free clique-triples before absorbing all 4-cliques into $I \cup S$, also we must maintain that every 4-clique not contained in $I \cup S$ intersects a free clique-triple.

These two properties hold initially. Suppose that they hold when we enter an instance of Step 3. We have mates $T_a$ and $T_b$, with $T_a$ being free. Since Step 2 does not apply, $b_3 \notin S$, so $T_b$ also is free. Since $G$ has no bonus 4-clique, $c \neq d$.

In the first case, $\{c_1, d_1, c_2, d_2\}$ is not a 4-clique, and we add $\{a_3, b_1\}$ to $I$. This adds $\{a_1, a_2, b_2, b_3\} \cup N_{G_1}(a_3)$ to $S$, gaining six vertices. The 4-clique $\{a_1, a_2, b_1, b_2\}$ has been absorbed. The vertices of other 4-cliques that might enter $I \cup S$ are those in $T_c \cup T_d$.

Suppose that $\{c_1, c_2, x_1, x_2\}$ is a 4-clique, with $T_x$ the mate of $T_c$. If $T_x$ is not free before this instance of Step 3, then $x_3 \in S$, but now Step 2 would have applied instead of Step 3, with $T_c$ as $T_a$ and $T_x$ as $T_b$. Since the addition to $I$ does not affect $x_3$, afterwards $T_x$ remains free. Similarly, the mate of $T_d$ remains free if $T_d$ is a clique-triple.

In the second case, $\{c_1, d_1, c_2, d_2\}$ is a 4-clique, and we add $\{c_3, d_1\}$ to $I$. This is an instance of the first case for the mates $T_c$ and $T_d$ unless $N_{G_1}(c_3) = \{a_3, b_3\}$. However, that requires $G = G_{DN}$, labeled as in Figure 1. Since $G \in \mathcal{G}'$, we find a 4-clique where the first case of Step 3 applies. □

**Theorem 2.8.** *For $G \in \mathcal{G}'$, using the output of Algorithm 2.6 as initialization to Algorithm 2.1 produces an independent set having more than $1/4$ of the vertices of $G$.*

*Proof.* By Lemma 2.2, we may assume that the output of Algorithm 2.6 is an independent set $I$ with neighborhood $S$ such that $3|I| = |S|$ and every 4-clique is contained in $I \cup S$. Furthermore, if $G$ has no 4-clique, then $I \cup S \neq V(G)$. To complete the proof, we show that with such an initialization, the final step of Algorithm 2.1 adds at most two vertices to $S$ (hence strict inequality holds at the end).

We claim that also when $G$ has a 4-clique and Algorithm 2.6 ends with $3|I| = |S|$, we have $I \cup S \neq V(G)$. We noted in the proof of Lemma 2.7 that ending in Step 1 or 2 yields $3|I| > |S|$, so ending with $3|I| = |S|$ requires ending in Step 3. On the last step, we have free mates $T_a$ and $T_b$, and we add $\{a_3, b_1\}$ to $I$ and $\{a_1, a_2, b_2, b_3\} \cup N_{G_1}(a_3)$ to $S$. If this exhausts $V(G)$, then before the final step we have $N_{G_1}(a_3) = V(G) - (I \cup S) - (T_a \cup T_b)$. The other vertices of the triples containing the vertices of $N_{G_1}(a_3)$ are already in $S$. These two vertices lie in the same triple; otherwise, each has at most two neighbors outside $S$ before the last step, and Step 1 would apply. On the other hand, if they belong to the same clique, then $\{a_1, a_2, b_1, b_2\}$ is a bonus 4-clique, which would have been used at the beginning.

Hence we may assume that at least one vertex remains outside $I \cup S$ when we move to Algorithm 2.1. We claim that at most two vertices are added to $S$ in the final step of Algorithm 2.1. If three vertices are added to $S$, then let $x$ be the vertex added to $I$, with neighbors $u, v, w$ added to $S$. Choosing one of $\{u, v, w\}$ instead of $x$ would also add at least three vertices to $S$, since we chose $v$ to minimize $|N(v) - S|$. This implies that $\{u, v, w, x\}$ is a 4-clique in $G$. This possibility is forbidden, since all vertices contained in 4-cliques are added to $I \cup S$ during Algorithm 2.2. □

**Corollary 2.9.** *Every 2-factor-plus-triangles graph has independence ratio at least $1/4$, with equality only for graphs whose components are all isomorphic to $G_{DN}$.*

## 3 Constructions

The Du-Ngo graph $G_{DN}$ is the only graph in $\mathcal{G}'$ with independence ratio 1/4. In this section, we construct an infinite sequence of graphs with independence ratio less than 4/15.

Figure 3 shows a 27-vertex graph $G$ in $\mathcal{G}'$ with $\alpha(G) = \frac{1}{4}(27+1)$. Note that $G$ is connected. An independent set $I$ has at most six vertices in the subgraph inside the dashed box (at most two from each "column" of 4-cycles). Also, $I$ has at most one vertex in the remaining 3-cycle $[x_3, y_3, z_3]$ in $G_1$. Hence $\alpha(G) \leq 7 = (27+1)/4$, and $\{a_1, b_3, c_1, d_3, e_1, f_3, x_3\}$ achieves the upper bound.

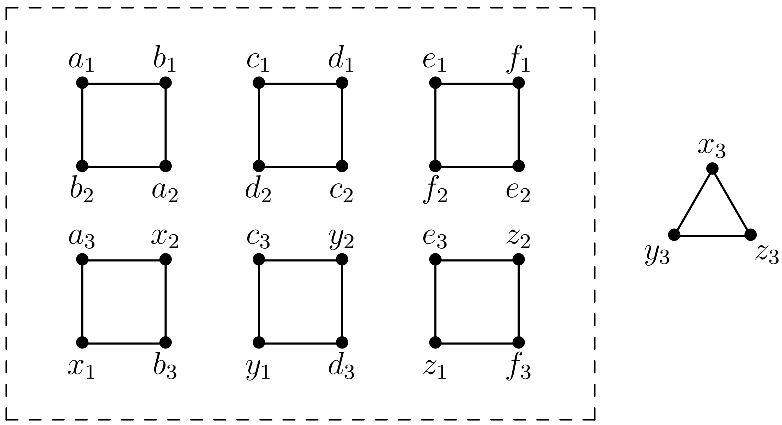


Figure 3: A graph in $\mathcal{G}'$ with independence number $(n+1)/4$

One may ask whether there are infinitely many graphs $G$ in $\mathcal{G}'$ with $\alpha(G) = (|V(G)|+1)/4$, or at least with $\alpha(G) \leq (|V(G)| + c)/4$ for some constant $c$. We conjecture that no such constant exists; in fact, we conjecture the following stronger statement.

**Conjecture 3.1.** *For every $t < 4/15$, only finitely many graphs in $\mathcal{G}'$ have independence ratio at most $t$.*

This conjecture is motivated by the following theorem, which shows that the conclusion is false when $t \geq 4/15$. To avoid confusion with our earlier use of $G_1$ and $G_2$, we use $Q_i$ and $R_i$ to index sequences of special graphs in this construction.

**Theorem 3.2.** *For $i \geq 0$, there is a graph $Q_i \in \mathcal{G}$ with independence ratio $\frac{4(2^i)-5/3}{15(2^i)-6}$.*

*Proof.* We first construct a rooted graph $R_i$ for $i \geq 0$. Then $Q_i$ will be built from three disjoint copies of $R_i$ by adding a 3-cycle on the roots. With $v$ denoting the root of $R_i$, let $R_i' = R_i - v$. We construct $R_i$ with $n_i$ vertices such that

1. $n_i = 15(2^i) - 6$ and $R_i$ is connected,

2. $R_i$ decomposes into a 2-factor on $R_i'$ and $n_i/3$ disjoint triangles, and
3. $\alpha(R_i') = 4(2^i) - 2$, with a maximum independent set avoiding the neighbors of $v$.

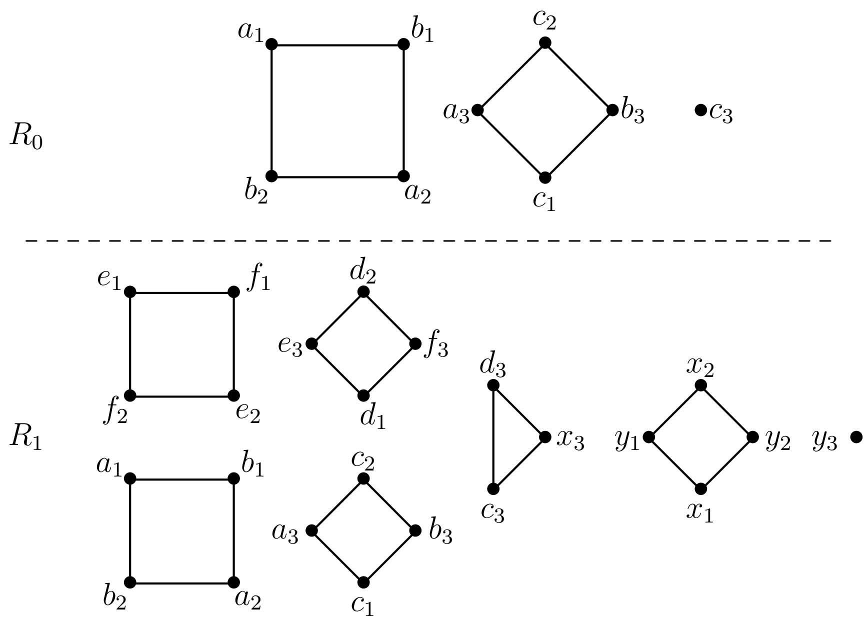


Figure 4: The graphs $R_0$ and $R_1$

We show $R_0$ in Figure 4 with root $c_3$. This graph is connected, has $15(2^0)-6$ vertices, and is the union of a 2-factor on $R_0'$ and triangles with vertex sets $T_a$, $T_b$, and $T_c$. An independent set in $R_0'$ has at most one vertex from each 4-clique, and $\{a_1, b_3\}$ is an independent set of size 2 avoiding $T_c$, so $\alpha(R_0') = 4(2^0) - 2 = 2$.

For $i \geq 1$, start with two disjoint copies of $R_{i-1}$, having roots $c_3$ and $d_3$. Add triples $T_x$ and $T_y$ on six new vertices. Augment the union of the 2-factors in the copies of $R_{i-1}'$ by adding the 3-cycle $[c_3, d_3, x_3]$ and the 4-cycle $[x_1, y_1, x_2, y_2]$. Leave $y_3$ as the root in the resulting graph $R_i$. Figure 4 shows $R_1$.

Doubling and adding six vertices shows inductively that $n_i = 15(2^i)-6$. By construction, $R_i$ is the union of a 2-factor on $R_i'$ and $n_i/3$ disjoint triangles. For connectedness, note that inductively each vertex in a copy of $R_{i-1}$ has a path to its root, and using the added 3-cycle, 4-cycle, and triples yields a path from each vertex to the root of $R_i$.

It remains to check property (3). Let $I$ be an independent set in $R_i'$. Maximizing the contributions to $I$ from the two copies of $R_{i-1}'$ yields $|I| \leq 2\alpha(R_{i-1}') + 2 = 4(2^i) - 2$. Furthermore, since $R_{i-1}'$ has a maximum independent set avoiding the neighbors of the root of $R_{i-1}$, we can use $c_3$ and $x_1$ as the two added vertices from $R_i'$, thereby forming a maximum independent set in $R_i'$ that avoids $T_y$.

In forming $Q_i$ by adding a 3-cycle on the roots of three disjoint copies of $R_i$, we obtain a connected 2-factor-plus-triangles graph. We can obtain maximum contribution from the three copies of $R_i'$ obtained by deleting the roots without using any neighbor of the roots. Hence $\alpha(Q_i) = 3\alpha(R_i') + 1 = 12(2^i) - 5$. With $Q_i$ having $3n_i$ vertices, we obtain the independence ratio claimed. □

In light of Erdős' question concerning the 3-colorability of graphs in $\mathcal{G}$ when 4-cycles are excluded from $G_1$, it is natural to ask whether this additional condition guarantees independence ratio 1/3. The answer is no. In fact, for every odd $g$ we construct infinitely many graphs in $\mathcal{G}'$ with independence ratio less than 1/3 formed using a 2-factor that has girth $g$. When $g \equiv 1 \mod 6$, the smallest graph in our family has $g^2 + 2g$ vertices; this suggests the following conjecture, which by our construction would be asymptotically sharp.

**Conjecture 3.3.** *Every $n$-vertex graph in $\mathcal{G}'$ with girth at least $\sqrt{n}$ has an independent set of size at least $n/3$.*

Our construction was motivated by an arrangement of triples on a 7-cycle, where two of the triples have one element off the cycle. This arrangement, shown in Figure 5, is due to Sachs (see [6]). We use it to build examples with girth 7. For larger $g$ congruent to 1 modulo 6, we construct an arrangement on a $g$-cycle. A special list allows us to enlarge the arrangement by multiples of 6.

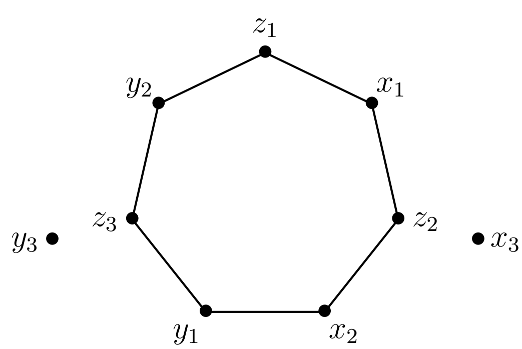


Figure 5: The graph $H_7'$

**Definition 3.4.** An *$a, b$-brick* is a list of six characters plus two holes called *notches*: $(a_1, \square, b_1, a_2, b_2, a_3, \square, b_3)$. An $a, b$-brick can link to a $c, d$-brick by starting the $c, d$-brick at the second notch in the $a, b$-brick. The last element of the $a, b$-brick then fits into the first notch in the $c, d$-brick. Such linkages leave notches in the second and next-to-last positions.

A *starter brick* is a list of seven characters plus two notches that has the form $(y_1, \square, y_2, z_1, x_1, z_2, x_2, \square, z_3)$. For $g = 6j + 1$, let $H_g'$ consist of two special vertices $x_3$ and

$y_3$ plus the cycle of length $g$ whose vertices in order are named by a cyclic arrangement having a starter brick and $a^{(i)}, b^{(i)}$-bricks for $1 \le i \le j-1$, linked together in order. The $a^{(1)}, b^{(1)}$-brick links to the second notch of the starter brick, and the $a^{(j-1)}, b^{(j-1)}$-brick links at its end to the first notch of the starter brick. In the degenerate case $j = 1$, the starter brick links to itself, producing the graph $H'_7$ shown in Figure 5. For each symbol $q$, the vertices of $\{q_1, q_2, q_3\}$ in $H'_g$ form a triangle. Note that $H'_g$ has $g + 2$ vertices.

The remaining theorems in this section rest on the following simple lemma.

**Lemma 3.5.** *Let $I$ be an independent set intersecting triples $T_a$ and $T_b$ in a graph $G$ in $\mathcal{G}$. If $T_a$ and $T_b$ form an $a, b$-brick in $G_1$, and $I$ contains the vertex in a notch of the $a, b$-brick, then $I$ also contains the vertex farthest from it in the $a, b$-brick.*

*Proof.* An $a, b$-brick has the form $(a_1, \square, b_1, a_2, b_2, a_3, \square, b_3)$. If $I$ contains the vertex in the first notch, then $I$ omits $a_1$ and $b_1$. Since $I$ must intersect $T_a$, we have $b_2 \notin I$. Hence $I$ must contain $b_3$ to intersect $T_b$. $\square$

**Theorem 3.6.** *For each odd $g$, there are in $\mathcal{G}'$ infinitely many graphs with girth $g$ whose independence ratio is less than $\frac{1}{3}$.*

*Proof.* First suppose that $g = 6j + 1$. For $k \ge 1$, we construct such a graph $H_{g,h,k}$ with $(g+2)hk$ vertices. Start with $hk$ copies of the graph $H'_g$ constructed in Definition 3.4, where $h$ is odd and at least 3. The vertices having the three subscripted copies of a given label form a triple, with $x_3$ and $y_3$ lying outside the cycle as in Figure 5. Each copy of $H'_g$ requires an additional superscript in the labels to distinguish its vertices from those of other copies.

Number the copies 0 through $hk - 1$. For $0 \le i \le k - 1$, add a cycle on the vertices representing $x_3$ in copies $hi + 1$ through $hi + h \pmod{hk}$ of $H'$, and add a cycle on the vertices representing $y_3$ in copies $hi$ through $hi + h - 1$ of $H'$. This completes the graph $H_{g,h,k}$; note that it has $(g + 2)hk$ vertices and is a 2-factor-plus-triangles graph.

Since $H'_g$ has an $x_3, y_3$-path, the cycles on the copies of $x_3$ and $y_3$ make it possible to reach each copy of $H'$ from any other. Hence $H_{g,h,k}$ is connected.

Each cycle in the 2-factor forming $H_{g,h,k}$ has length $g$ or length $h$. A cycle of length $h$ contributes at most $(h - 1)/2$ vertices to an independent set; we apply this to the cycles through the copies of $x_3$ and $y_3$. There are $2k$ of these cycles, so they contribute at most $k(h - 1)$ vertices. In addition, we claim that the $g$-cycle in each copy of $H'_g$ contributes at most $2j$ vertices to an independent set; note that $2j = (g - 1)/3$. If this claim is true, then

$$\alpha(H_{g,h,k}) \le hk\frac{g-1}{3} + k(h-1) = hk\frac{g+2}{3} - k < hk\frac{g+2}{3} = \frac{1}{3}|V(H_{g,h,k})|.$$

The inequality would be too weak if the $g$-cycle could contribute $2j + 1$ vertices.

To prove the claim, note that the $g$-cycle contains the vertices of $2j-1$ full triples (including one in the starter brick) plus $\{x_1, x_2, y_1, y_2\}$. To contribute more than $2j$ vertices, we must find an independent set having an element from each full triple, plus one of $\{x_1, x_2\}$ and one of $\{y_1, y_2\}$.

Suppose that such an independent set $I$ exists. Since the last vertex of each brick fits into the first notch of the next brick, $z_3 \in I$ implies $b_3^{(j-1)} \in I$, and $y_1 \in I$ implies $a_1^{(1)} \in I$, by applying Lemma 3.5 iteratively to each ordinary brick. In the first case, $b_3^{(j-1)} \in I$ forbids having a vertex from $\{y_1, y_2\}$. In the second case, $x_2, z_3 \notin I$, and $I$ cannot have two elements in $\{z_1, x_1, z_2\}$. Both arguments apply in degenerate form when $k = 0$.

In the remaining case, $z_3, y_1 \notin I$. Here one from each of $T_x, T_y, T_z$ must be chosen nonconsecutively from the string $(y_2, z_1, x_1, z_2, x_2)$, and this is not possible. This completes the argument for $g \equiv 1 \mod 6$.

When $g \not\equiv 1 \pmod 6$, we set $h$ to be $g$ and let the first value higher than $g$ that is congruent to 1 modulo 6 play the role of $g$ in the construction above. Since $k$ is arbitrary, the family is still infinite. □

To form the smallest example in the construction of Theorem 3.6 when $g \equiv 1 \mod 6$, set $h = g$ and $k = 1$. The resulting graph $H_{g,g,1}$ has girth $g$ and has $g^2 + 2g$ vertices. Letting $n = |V(H_{g,g,1})|$, we have an $n$-vertex example where $G_1$ has girth $\sqrt{n+1} - 1$ and the independence ratio (of $H_{g,g,1}$) is less than $1/3$. When $g \not\equiv 1 \mod 6$ and we are forced to use $H'_{g'}$ for some $g'$ larger than $g$, we use even more vertices. This motivates Conjecture 3.3.

Although girth at least $\sqrt{n}$ in $G_1$ may be enough to force an independent set of size $n/3$ in $G$, it does not force $G$ to be 3-colorable. Surprisingly, there is no threshold for the girth in terms of $n$ that forces this except $n$ itself, where $G$ becomes a cycle-plus-triangles graph. Note that if the girth of an $n$-vertex 2-regular graph $G_1$ is not $n$, then it is at most $n/2$.

**Theorem 3.7.** *If $n = 24 + 12k$ with $k \geq 0$, then there is an $n$-vertex 2-factor-plus-triangles graph $G$ such that $G_1$ consists of two $n/2$-cycles and $G$ is not 3-colorable.*

*Proof.* We use $a^{(i)}, b^{(i)}$-bricks as in Theorem 3.6, but for this theorem the starter bricks have 12 symbols plus two notches. We use two starter bricks:

$$(z_1, \square, z_2, u_1, z_3, u_2, v_3, w_3, y_2, x_3, y_1, x_2, \square, y_3)$$

$$(\hat{z}_2, \square, \hat{z}_3, v_1, w_1, \hat{z}_1, v_2, w_2, u_3, \hat{y}_2, x_1, \hat{y}_3, \square, \hat{y}_1)$$

Let $G_1$ consist of cycles $C$ and $\hat{C}$, where $C$ consists of the first starter brick and $a^{(i)}, b^{(i)}$-bricks for $1 \leq i \leq k$, and $\hat{C}$ consists of the second starter brick and $\hat{a}^{(i)}, \hat{b}^{(i)}$-bricks for $1 \leq i \leq k$, linked in order as in Theorem 3.6. The triples for $u, v, w, x$ create connections between the two cycles, but all other triples are confined to $C$ or to $\hat{C}$. When $k = 0$, each starter brick links into itself to form a 12-cycle. (Examples with $n$ vertices and girth $n/2 - 6r$ arise by using $k - r$ ordinary bricks in $C$ and $k + r$ ordinary bricks in $\hat{C}$; the same argument applies.

Suppose that the resulting graph $G$ has a proper 3-coloring $f$. Each color class is an independent set having one vertex in each triple. Simplifying notation, let $b_3$ and $a_1$ denote the vertices in the first and second notches of the starter brick in $C$, respectively, while $\hat{b}_3$ and $\hat{a}_1$ denote those vertices in $\hat{C}$. Without loss of generality, we may assume that $f(a_1) = 1$. Repeatedly applying Lemma 3.5 yields $f(z_1) = 1$. Now we may assume that $f(b_3) = 3$; repeatedly applying Lemma 3.5 yields $f(y_3) = 3$.

If the neighbors in $G_1$ of a vertex $\alpha$ belong to the same triple, then the third member of that triple must have the same color as $\alpha$. Hence $f(x_3) = f(y_3) = 3$, and $f(u_1) = f(z_1) = 1$. Also, if a vertex next to $\alpha$ and another member of the triple containing $\alpha$ have distinct colors, then $f(\alpha)$ is the third color. Hence $f(x_2) = 2$ and $f(z_2) = 2$. Once we color two members of a triple, the third has the third color. Hence $f(x_1) = 1$ and $f(z_3) = 3$. If two neighbors of $\alpha$ have distinct colors, then $\alpha$ has the third color. Hence $f(y_1) = 1$. Now $f(y_2) = 2$.

Since $f(z_3) = 3$ and $f(u_1) = 1$, we have $f(u_2) = 2$, and then $f(u_3) = 3$. Now $f(x_1) = 1$ and $f(u_3) = 3$ imply $f(\hat{y}_2) = 2$, and hence $f(\hat{y}_3) = 3$ and $f(\hat{y}_1) = 1$. This leaves $f(\hat{a}_1) = 2$. Iterating Lemma 3.5 now yields $f(\hat{z}_2) = 2$ and $f(\hat{b}_3) = 1$. Now $f(\hat{z}_3) = 3$ and $f(\hat{z}_1) = 1$.

We have now determined the colors of all vertices in the starter bricks except those in the triples $T_v$ and $T_w$. For all other vertices in these bricks, the color matches the subscript. The relevant remaining segments are $(u_2, v_3, w_3, y_2)$ and $(\hat{z}_3, v_1, w_1, \hat{z}_1, v_2, w_2, u_3)$. Color 2 is forbidden from $\{v_3, w_3\}$. Hence it appears on one of $\{v_1, v_2\}$ and one of $\{w_1, w_2\}$. However, the subscripts on its appearances differ. If $f(v_1) = f(w_2) = 2$, then $f(w_1) = f(v_2) = 3$ (since $f(\hat{z}_1) = 1$), and then $f(v_3) = f(w_3)$. If $f(v_2) = f(w_1) = 2$, then $f(w_2) = f(v_1) = 1$ (since $f(\hat{z}_3) = f(u_3) = 3$), and again $f(v_3) = f(w_3)$. Hence the coloring cannot be completed. □

# 4 Triangles-Plus-Triangles Graphs

Although some 2-factor-plus-triangles graphs are not 3-colorable, some (such as cycle-plus-triangles graphs) are 3-choosable. Another such class occurs at the other "extreme", when the cycles in the 2-factor are 3-cycles. That is, the union of two graphs on the same vertex set whose components are all triangles is 3-choosable.

We prove a more general statement in terms of the numbers of vertices in the components of two subgraphs whose union is $G$. Our main tool is the theorem of Galvin [7] about list coloring of the line graphs of bipartite graphs: if $G$ is a bipartite multigraph with maximum degree $k$, then the line graph of $G$ is $k$-choosable.

**Proposition 4.1.** *If $G_1$ and $G_2$ are graphs whose components have at most $s$ vertices, then $G_1 \cup G_2$ is $s$-choosable.*

*Proof.* Let $G = G_1 \cup G_2$. By adding isolated vertices to $G_1$ and/or $G_2$ as needed, we may assume that $V(G_1) = V(G_2) = V(G)$ without changing $G$. For each $v \in V(G)$, let $L(v)$ be a set of $s$ available colors. Form a graph $H$ with one vertex for each component of $G_1$ and one vertex for each component of $G_2$. For each vertex of $G$, place an edge in $H$ joining the vertices representing the components containing it in $G_1$ and $G_2$ ($H$ is the "intersection graph" of the components in $G_1$ and $G_2$). By construction, $H$ is bipartite. The degree of a vertex in $H$ is the number of vertices in the corresponding component of $G_1$ or $G_2$.

Each edge of $H$ corresponds to a vertex $v$ in $G$. Assign to this edge the list $L(v)$. Since $H$ is bipartite and has maximum degree at most $s$, Galvin's Theorem implies that we can choose a proper edge-coloring of $H$ from the lists. This assigns colors to the vertices of $G$ from their lists so that vertices in the same component of $G_1$ or in the same component of $G_2$ have distinct colors. Hence it is a proper coloring of $G$. □

In particular, every triangles-plus-triangles graph is 3-choosable.

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