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A Length–Width Inequality for Partially Ordered Sets with Two-Element Cutsets

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We show that if P is a partially ordered set of width n, and A is an antichain of size n whose elements all have cutsets of size at most 2, then every maximal chain of P has at least $n-2$ elements. We also give an extension to larger cutsets. © 1989 Academic Press, Inc.

1. INTRODUCTION

Recall that the *width* $w(P)$ of a finite partially ordered set (poset) P is the size of the largest antichain in P. We let $l(P)$ denote the number of elements in the largest chain in P. If a and b are elements of a poset P and a is incomparable to b, we write $a \parallel b$. For an element $a \in P$ and a subset S of P we write $a \parallel S$ to denote that $a \parallel x$ for all $x \in S$ and we write $a \leqslant S$ to denote that $a \leqslant x$ for all $x \in S$. The number of elements in a set X is denoted by $|X|$.

Let x be an element of a poset P. A *cutset* for x is a set C of elements of P such that (i) $x \parallel C$ and (ii) every maximal chain of P intersects $C \cup \{x\}$. We say P has the *m-cutset property* if every element of P has a cutset with at most m elements.

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In this paper we prove:

THEOREM 1. *If P is a finite poset with the 2-cutset property, then* $l(P) \geqslant w(P) - 2$.

Actually, we prove a stronger result. We get a lower bound of $l(P) \geqslant w(P) - 2$ even without the 2-cutset property, as long as P has a maximum-sized antichain whose elements have cutsets of size at most 2. In fact, part (b) of the following theorem gives the stronger conclusion that in this case every maximal chain of P has at least $w(P) - 2$ elements. Part (a) of the theorem, which also is stronger than Theorem 1, guarantees $l(P) \geqslant n - 2$ whenever P has an n-element antichain satisfying a somewhat weaker condition implied by its elements having cutsets of size 2.

THEOREM 2. *Let P be a poset containing an antichain $A = \{a_1, ..., a_n\}$, and let $A_1, ..., A_n$ be subsets of P such that for each i, $|A_i| \leqslant 2$ and $a_i \| A_i$.*

(a) *If for each i, each maximal chain in P containing a_i intersects every A_j for $j \neq i$, then every maximal chain in P intersecting A has at least $n - 2$ elements.*

(b) *If A is a maximal antichain of P and each A_i is a cutset for a_i, then every maximal chain in P has at least $n - 2$ elements.*

The poset of Fig. 1 shows that both theorems are best possible. Theorem 1 has also been established by M. El-Zahar and N. Sauer [2], using a different approach.

The idea of a cutset originated with Bell and Ginsburg [1] in connection with a question in topology. This concept has since been explored in purely order-theoretic contexts by Ginsburg *et al.* [3] and by Sauer and Woodrow [4]. For instance, Sauer and Woodrow show that if P is a poset with the 2-cutset property then every element of P belongs to a maximal antichain of at most 4 elements. (Note that the definition of the m-cutset

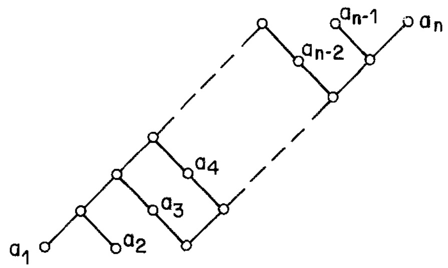

FIG. 1. A poset of width n with the 2-cutset property, in which every maximal chain has $n - 2$ elements.

property in [2, 4] differs slightly from ours. In [2, 4] it is that every element has a cutset containing *fewer than m* elements.)

In the next section we introduce some notation and prove the lemmas we need for Theorem 2, which is proved in Section 3. In Section 4 we extend Theorem 2(a) by a simple induction to the case where each set A_i has size at most m ($m > 2$). This results in the following inequality relating $l = l(P)$ and the width $w(P)$ of a finite poset with the m-cutset property:

$$w(P) \leqslant \frac{(l^2 + l - 1)l^{m-2} - 1}{l - 1}.$$

For $m = 2$, we obtain $w(P) \leqslant l(P) + 2$, which is Theorem 1. Although the inequality is sharp for $m = 2$, we do not expect it to be close to best possible for larger m.

2. The Graph $G(A)$ and Preliminary Lemmas

For this section we assume the following situation: $A = \{a_1, ..., a_n\}$ is an antichain of a poset P, and $A_1, ..., A_n$ are subsets of P such that for each i, (i) $|A_i| \leqslant 2$, (ii) $a_i \| A_i$, and (iii) every maximal chain in P containing a_i intersects each A_j for $j \neq i$. These are precisely the conditions of part (a) of Theorem 2 and they are implied by the condition of part (b). Thus the results of this section are valid under the hypothesis of *either* part (a) or part (b) of Theorem 2.

Define a graph $G(A)$ as follows: the vertices of $G(A)$ are the subsets $A_1, A_2, ..., A_n$, and two subsets A_i and A_j are adjacent in $G(A)$ if and only if they intersect.

Lemma 1. *$A_i = A_j$ implies $i = j$. Thus, A_i and A_j are adjacent in $G(A)$ iff $|A_i \cap A_j| = 1$.*

Proof. If, say, $A_1 = A_2$ then a maximal chain through a_1 cannot contain a_2 and so must contain some element from $A_2 = A_1$. But this contradicts the fact that no element in A_1 is comparable with a_1. ∎

It is easy to prove (e.g., [3, 4]) that every poset with the 1-cutset property has width at most two. A very similar proof yields:

Lemma 2. *For distinct i, j, k, $A_i \cap A_j \cap A_k = \varnothing$. In particular, every vertex of $G(A)$ has degree at most 2.*

Proof. Otherwise, suppose $A_i = \{b, c_i\}$ for $i = 1, 2, 3$. Since a_1 is incomparable with a_2, a_3, and b, any maximal chain through a_1 must contain c_2 and c_3. Therefore c_2 and c_3 are comparable, and by symmetry $\{c_1, c_2, c_3\}$

is a chain. Without loss of generality, $c_1 < c_2 < c_3$. Since a_1 and c_2 are comparable but a_1 and c_1 are not, we must have $a_1 < c_2$; similarly $a_3 > c_2$. But this contradicts the fact that A is an antichain. ▮

Therefore $G(A)$ is a disjoint union of cycles, paths, and isolated points. Since every maximal chain in P containing a_i must contain an element from every other A_j, and since two of the A_j's have an element in common only if they are adjacent in $G(A)$, this shows that every maximal chain in P containing an element of A must contain roughly $n/2$ elements at least. The essence of our proof, the part that lets us push $n/2$ up to $n-2$, is that each chain meeting A contains at most two common elements of the A_j's doing double duty. This follows from a sequence of lemmas about such common elements.

We will find it useful to employ the following terminology and notation relating to edges in $G(A)$. If $A_iA_{i'}$ is an edge, then A_i and $A_{i'}$ intersect in a single element $b_{ii'}$, which we call a *bond*, and we write $A_i = \{b_i, b_{ii'}\}$ and $A_{i'} = \{b_{ii'}, b_{i'}\}$. We have $a_i \notin \{b_i, b_{ii'}\}$ by hypothesis, and we also know $b_i \neq b_{i'}$ and $b_{ii'} \notin A_j$ for $j \notin \{i, i'\}$ by Lemmas 1 and 2, but in other cases we may have $b_i \in A_j$ for $j \notin \{i, i'\}$ or $a_i \in A_j$ for $j \neq i$.

If C is a chain in P and $x \in P$, we will say that x is *on the end of* C if $x \leqslant C$ or $C \leqslant x$. We then say that x is on the *bottom* or *top* of C, respectively. If $(c_1, ..., c_n)$ is an arbitrary listing of the elements of a chain C, we let $\mathscr{C}(c_1, c_2, ..., c_n)$ denote the family of all maximal chains of P containing $\{c_1, c_2, ..., c_n\}$. Let

$$\overline{c_1 c_2 \cdots c_n} = \bigcap \mathscr{C}(c_1, c_2, ..., c_n)$$

and

$$\langle c_1 c_2 \cdots c_n \rangle = \bigcup \mathscr{C}(c_1, c_2, ..., c_n).$$

Thus $x \in \overline{c_1 c_2 \cdots c_n}$ means that x is in every maximal chain of P containing $\{c_1, c_2, ..., c_n\}$, while $x \in \langle c_1 c_2 \cdots c_n \rangle$ means that $\{x, c_1, ..., c_n\}$ is a chain. Note that

$$\overline{\overline{c_1 c_2 \cdots c_n}} = \overline{c_1 c_2 \cdots c_n}.$$

LEMMA 3. *Let $A_iA_{i'}$ be an edge of $G(A)$. Then every maximal chain containing a_i must contain $b_{i'}$ (in short, $b_{i'} \in \overline{a_i}$).*

Proof. Any maximal chain containing a_i must contain an element of $A_{i'}$. But $b_{ii'} \in A_i$ is incomparable to a_i, so $b_{i'} \in \overline{a_i}$. ▮

LEMMA 4. *Let $A_iA_{i'}$ and $A_jA_{j'}$ be two independent edges in $G(A)$. Then a_i cannot lie on the end of a chain which contains b_j and $b_{j'}$.*

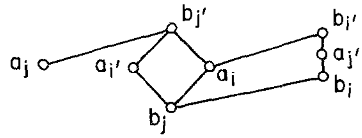

FIGURE 2

Proof. By contradiction. By duality, suppose $a_i \leqslant b_j \leqslant b_{j'}$. Now, by Lemma 3, $a_{j'}$ is comparable to b_j. But $a_{j'} \leqslant b_j$ implies that $a_{j'} \leqslant b_{j'}$, and $a_{j'} \geqslant b_j$ implies that $a_{j'} \geqslant a_i$, both of which are false. ∎

LEMMA 5. *Let $A_iA_{i'}$ and $A_jA_{j'}$ be two independent edges in $G(A)$. If b_j and $b_{j'}$ are comparable, it is impossible that both b_j and $b_{j'}$ are comparable to both a_i and $a_{i'}$.*

Proof. Otherwise, by Lemma 4, we may assume $b_j \leqslant a_i \leqslant b_{j'}$ and $b_j \leqslant a_{i'} \leqslant b_{j'}$. By Lemma 3 we have $a_{j'} \geqslant b_j$ and $a_j \leqslant b_{j'}$, and in fact $\overline{a_{j'}} = \overline{b_j a_{j'}}$. Now consider how chains through $a_{j'}$ meet A_i and $A_{i'}$. We claim $\{b_i, b_{i'}\} \not\subset \overline{b_j a_{j'}}$. Otherwise, we may assume $b_i \leqslant b_{i'}$, which implies by Lemma 3 that $b_{i'} \geqslant a_i$, and by Lemma 4 that $b_{i'} \geqslant a_{j'} \geqslant b_i$. The situation at this point is show in Fig. 2. (In this figure, not all elements show need be distinct, nor are all incomparabilities certain.) By Lemma 3, $b_{i'} \in \overline{b_j a_i b_{j'}}$. Now we get a forbidden comparability by $b_{i'} \geqslant b_{j'} \geqslant a_{i'}$ or by $b_{j'} \geqslant b_{i'} \geqslant a_{j'}$.

Hence $\{b_i, b_{i'}\} \not\subseteq \overline{b_j a_{j'}}$. Since any maximal chain containing $a_{j'}$ must intersect both A_i and $A_{i'}$ it follows that $b_{ii'} \in \langle b_j a_{j'} \rangle$. Similarly, $b_{ii'} \in \langle a_j b_{j'} \rangle$. But now from $b_{ii'} \| a_i$ we get $b_{ii'} \geqslant b_j$, and similarly $b_{ii'} \leqslant b_{j'}$. Thus $b_{ii'} \geqslant a_j$, since $a_j \| b_j$ and $b_{ii'} \in \langle a_j, b_{j'} \rangle$. Similarly $b_{ii'} \leqslant a_{j'}$, and this contradicts $a_j \| a_{j'}$. ∎

LEMMA 6. *Let $A_jA_{j'}$ and $A_kA_{k'}$ be two independent edges in $G(A)$ and let A_i be a vertex of $G(A)$ belonging to neither. Then a_i cannot lie on the end of a chain which contains $b_{jj'}$ and $b_{kk'}$.*

Proof. Otherwise, by symmetry we can suppose that $a_i \leqslant b_{jj'} \leqslant b_{kk'}$. It follows that $b_{jj'} \| \{a_k, a_{k'}\}$, since $a_i \| \{a_k, a_{k'}\}$ and $b_{kk'} \| \{a_k, a_{k'}\}$. Therefore $\{b_j, b_{j'}\} \subseteq \overline{a_k} \cap \overline{a_{k'}}$ which is impossible by Lemma 5. ∎

3. THE PROOF OF THEOREM 2

Consider a maximal chain C intersecting A. Meeting every other A_j forces C to have at least n elements, except that elements can do "double duty" in two ways: (a) the element $a_i = C \cap A$ can be in some A_j, or (b) an element of C can be in two (no more) of the A_j's, if they are adjacent in $G(A)$. If a_i is not of type (a), then the number of elements that can be saved by hitting two of the A_j's at once (i.e., by containing bonds) is at most the maximum number of independent edges in $G(A)$. By Lemma 6, there can be at most one bond above and below a_i on a chain, so there are at least $n-2$ elements on any maximal chain through an a_i that is not in some A_j.

Lemma 6 also applies when a_i is of type (a), but does not yield the desired bound quite so easily. If a_i appears in exactly one other A_j, a chain through a_i may still contain two independent bonds (but no more, by Lemma 6) to get down to $n-3$ elements. If $a_i = b_{jj'}$ itself is a bond, we may get a chain with only $n-3$ or $n-4$ elements by using a_i and one or two other bonds for edges that form a matching in $G(A)$ with $A_jA_{j'}$.

In each of these cases we can choose indices so that the troublesome chain contains $\{a_5, b_{12}, b_{34}\}$, where b_{12}, b_{34} are bonds for independent edges. By symmetry, we have three cases for locating a_5: $a_5 = b_1 \in A_1$, $a_5 = b_{12}$, and $a_5 \in A_6$. We cannot have $a_5 = b_1$ because Lemma 3 implies $b_1 \in \overline{a_2}$. If $a_5 = b_{12}$ we must have $\{b_1, b_2\} \in \overline{a_3} \cap \overline{a_4}$, which violates Lemma 5. Hence we are reduced to the case $a_5 \in A_6$.

Letting $A_6 = \{a_5, c_6\}$, we will get a contradiction whether or not a_5 is a bond. By Lemma 6, we may assume the chain (a_5, b_{12}, b_{34}) is $b_{12} < a_5 < b_{34}$. We know that $c_6 \in \overline{a_1} \cap \overline{a_2} \cap \overline{a_3} \cap \overline{a_4}$, say with $c_6 \leqslant a_i$, $i = 1, ..., 4$. If $a_6 \parallel b_{12}$, then $b_1, b_2 \in \overline{a_6}$, and by symmetry we may assume $b_1 < b_2$. Lemma 3 implies $b_1 \in \overline{a_6} \cap \overline{c_6 a_2}$, which with $b_1 < b_2$ forces $b_1 \leqslant a_2$, then $b_1 \leqslant a_6$, and then $b_1 \leqslant c_6 \leqslant a_1$, a contradiction. Since we cannot have $a_6 \leqslant b_{12} < a_5$, the contradiction forces $a_6 > b_{12}$. But now $a_6 \parallel c_6$ and $a_2 \parallel b_{12}$ force $c_6 \parallel b_{12}$, so $\{b_1, b_2\} \subseteq \overline{c_6 a_3} \cap \overline{c_6 a_4}$, which contradicts Lemma 5. This completes the proof that any maximal chain through an element of A has at least $n-2$ elements.

Now consider a maximal antichain $A = \{a_1, ..., a_n\}$ and suppose that there is a cutset of size 2 for each $a_i \in A$. These cutsets A_i satisfy the conditions of part (a), so every maximal chain meeting A has at least $n-2$ elements. Any maximal chain C not meeting A intersects each A_i. Therefore, C must have at least $n-2$ elements unless it contains bonds for three independent edges in $G(A)$. Let b_{12} be the middle of three bonds on C. We have $b_{12} \parallel \{a_1, a_2\}$ by definition, and $b_{12} \parallel \{a_3, ..., a_n\}$ by Lemma 6, since any a_i related to b_{12} would be on the end of a chain with it and the other bond in C above or below. Hence $b_{12} \parallel A$, and we can include b_{12} to obtain a larger antichain.

We should note that the hypothesis of part (b) that A be a maximal antichain can be dropped. In particular, if P has an antichain of size n for which every element has a cutset of size 2, then every maximal chain of P has at least $n-2$ elements. As is clear from the proof above, it suffices to show that $G(A)$ cannot contain three independent edges. This fact is a

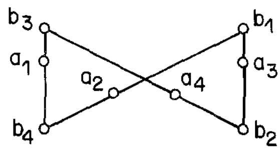

FIGURE 3

technical lemma whose proof involves a lengthy case analysis and is available from the authors.

Forbidding three independent edges implies that $G(A)$ cannot contain a cycle or path on more than five vertices. The lemma can be extended to forbid a 5-cycle, also. The unique minimal poset for which $G(A)$ is a 4-cycle is given in Fig. 3, with $A_i = \{b_i, b_{i+1}\}$ for $i \leqslant 3$ and $A_4 = \{b_4, b_1\}$. An example of a poset where $G(A)$ is a path on five vertices is given in Fig. 4, with $A_i = \{b_i, b_{i+1}\}$.

4. LARGER CUTSETS

To close, we consider an extension of Theorem 2 to posets with the m-cutset property for $m > 2$. Fix a positive integer l. Let $w(l, m)$ be maximal such that there exists a poset P with $l(P) = l$ containing an antichain $\{a_1, ..., a_{w(l,m)}\}$ and subsets $A_1, ..., A_{w(l,m)}$ with the following properties: for each i,

(i) $|A_i| \leqslant m$;

(ii) $a_i \| A_i$;

(iii) every maximal chain in P containing a_i intersects A_j for each $j \neq i$.

Clearly $w(l, m)$ is an upper bound for the width of any poset P with $l(P) = l$ and with the m-cutset property. Part (a) of Theorem 2 (together with Fig. 1) shows that $w(l, 2) = l + 2$. We remark also that $w(l, 1) = 2$, as can be seen from the proof of Lemma 2.

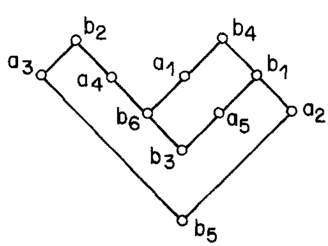

FIGURE 4

A simple pigeonhole argument yields:

THEOREM 3. *$w(l, m) \leqslant l \cdot w(l, m-1) + 1$. Thus $w(l, m) \leqslant ((l^2 + l - 1) l^{m-2} - 1)/(l-1)$ for $m \geqslant 2$.*

Proof. Set $n = w(l, m)$. Let P be a poset with $l(P) = l$ and containing an antichain $\{a_1, ..., a_n\}$ and subsets $A_1, ..., A_n$ satisfying (i)–(iii) above. Let C be a maximal chain containing a_n, say. C has at most l elements and must intersect each of the sets $A_1, ..., A_{n-1}$; thus there is some $x \in C$ which is contained in at least $(n-1)/l$ of the A_i's. Without loss of generality, $x \in A_1 \cap \cdots \cap A_t$ where $t \geqslant (n-1)/l$. Let $A_i' = A_i - \{x\}$ for $1 \leqslant i \leqslant t$. Then $|A_i'| \leqslant m-1$ for each $i \in \{1, ..., t\}$, and every maximal chain containing a_i must intersect A_j' for each $j \neq i$ in $\{1, ..., t\}$; for otherwise, by (iii) some maximal chain containing a_i must contain x, which is impossible by (ii) since $x \in A_i$. Thus

$$\frac{n-1}{l} \leqslant t \leqslant w(l, m-1),$$

which implies the first part of Theorem 3. Hence $w(l, m)$ exists for all m, and the upper bound for $w(l, m)$ follows from $w(l, 2) = l + 2$ by induction. ∎

COROLLARY. *If P is a finite poset with the m-cutset property, and $l(P) = l$, then*

$$w(P) \leqslant \frac{(l^2 + l - 1) l^{m-2} - 1}{l-1}.$$

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REFERENCES

1. M. BELL AND J. GINSBURG, Compact spaces and spaces of maximal complete subgraphs, *Trans. Amer. Math. Soc.* **283** (1984), 329–338.
2. M. EL-ZAHAR AND N. SAUER, The length, the width and the cutset-number of finite ordered sets, *Order* **2** (1985), 243–248.
3. J. GINSBURG, I. RIVAL, AND B. SANDS, Antichains and finite sets that meet all maximal chains, *Canad. J. Math.* **38** (1986), 619–632.
4. N. SAUER AND R.E. WOODROW, Finite cutsets and finite antichains, *Order* **1** (1984), 35–46.