1,2,3-Conjecture and 1,2-Conjecture for Sparse Graphs

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slides available on DBW preprint page

Joint work with Daniel W. Cranston and Sogol Jahanbekam

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Aim: Find proper S-weightings or total S-weightings with S being a small set of positive integers.

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The original conjectures are the special cases where the lists consist of the smallest positive integers.

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Graphs with maximum degree 3 are
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This talk: Explanation for $\mathrm{Mad}(G) < 5/2$.

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Vertices of degree 1 are not immediately reducible.

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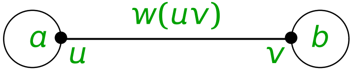

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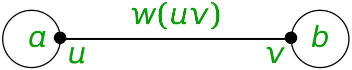

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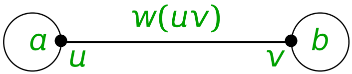

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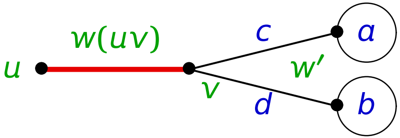

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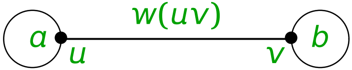

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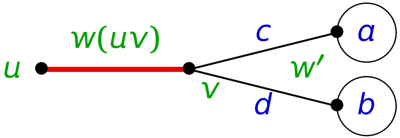

Extend w': pick $w(uv)$ to satisfy other edge(s) at v.
$w(uv) \in [3] - \{a - d, b - c\}$; always uv is satisfied. ■

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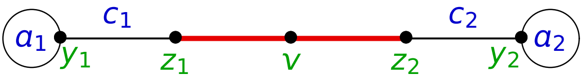

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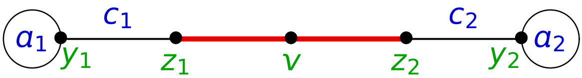

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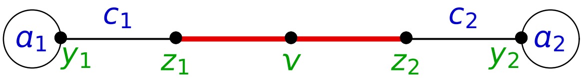

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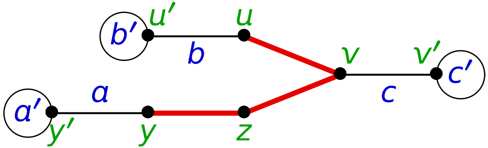

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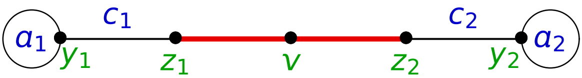

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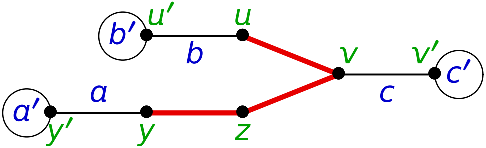

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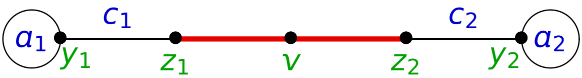

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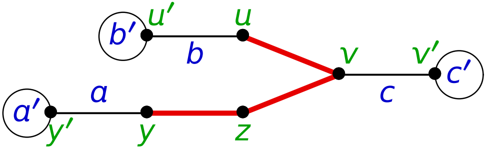

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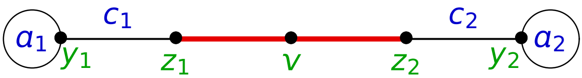

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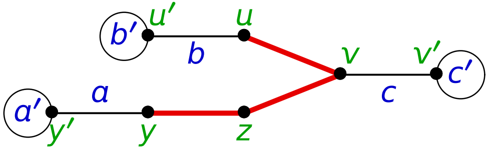

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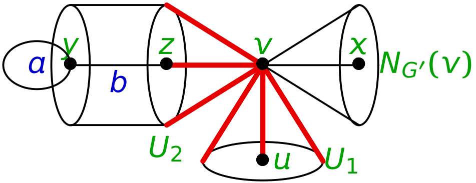

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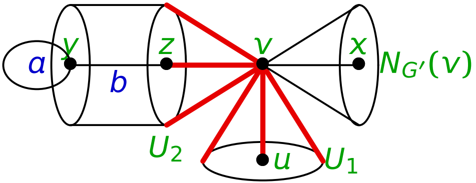

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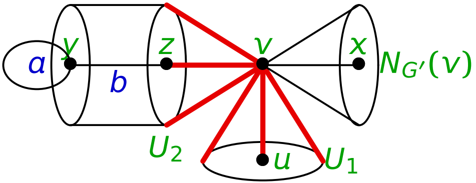

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Now let σ be the sum for new (red) edges. We need $\sigma \neq \phi_{w'}(x) - \phi_{w'}(v)$ for $x \in N_{G'}(v)$. Three choices in $[v, U_1]$ plus two in $[v, U_2]$ give $1 + 2p_1 + p_2$ choices for σ. Hence $2p_1 + p_2 \geq d_G(v) - p_1 - p_2$ suffices.

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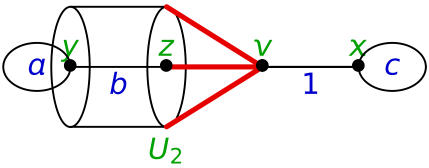

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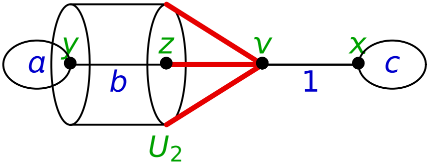

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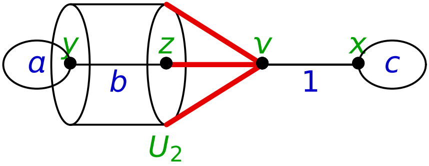

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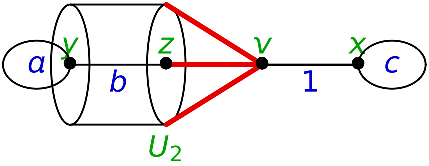

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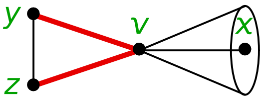
Isolated edges: To apply induction hypothesis, we must say what to do when G' has isolated edges.



Discharging Argument

Lem. If $\text{Mad}(G) < 5/2$ and G has no isolated edge, then G has one of these structures:

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Let $\mu(v)$ denote the resulting charge at v.
Claim: If G has no **A,B,C,D**, then $\mu(v) \geq \frac{5}{2}$ for all v.

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Excluded configurations:

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$d(v) \geq 5$: no **D** $\Rightarrow$
$\mu(v) = d(v) - \frac{1}{2}(3p_1+p_2) \geq d(v) - \frac{1}{2}(3p_1+2p_2) > \frac{d(v)}{2} \geq \frac{5}{2}$.

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Lem. The following configurations are $1, 2$-reducible:

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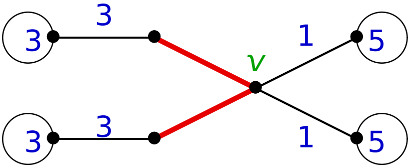

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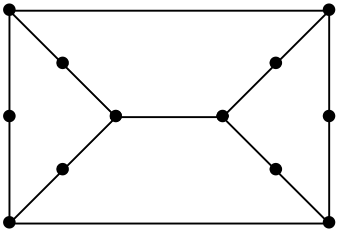

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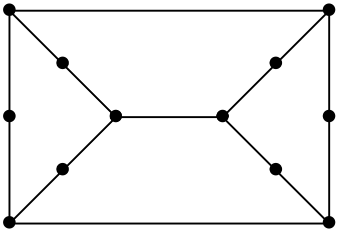

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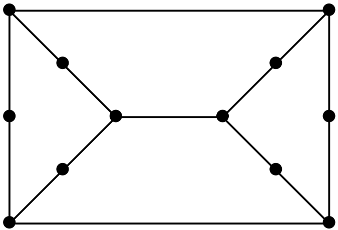

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Call such vertices b-vertices.

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$\text{Mad}(G) \geq 5/2$ does not force any configuration among the earlier $1, 2, 3$-reducible **A,B,C,D**, but avoiding **A,B,C,D** when $\text{Mad}(G) < 8/3$ requires *b*-vertices, which become $1, 2, 3$-reducible with just a bit of nearby sparseness. For example,

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On graphs with $\text{Mad}(G) < 8/3$, the proof is easier for the $1, 2$-Conjecture than the $1, 2, 3$-Conjecture.