

# The 1, 2, 3-Conjecture and 1, 2-Conjecture for Sparse Graphs

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## Abstract

The 1, 2, 3-Conjecture states that the edges of a graph without isolated edges can be labeled from  $\{1, 2, 3\}$  so that the sums of labels at adjacent vertices are distinct. The 1, 2-Conjecture states that if vertices also receive labels and the vertex label is added to the sum of its incident edges, then adjacent vertices can be distinguished using only  $\{1, 2\}$ . We show that various configurations cannot occur in minimal counterexamples to these conjectures. Discharging then confirms the conjectures for graphs with maximum average degree less than  $8/3$ . The conjectures are already confirmed for larger families, but the structure theorems and reducibility results are of independent interest.

*Keywords:* 1, 2, 3-Conjecture, 1, 2-Conjecture, reducible configuration, discharging method.

*Mathematics Subject Classification:* 05C15, 05C22, 05C78.

## 1 Introduction

Coloring problems in graph theory consider many ways of generating vertex colorings. We consider restricted colorings produced from weights on the edges and vertices.

Let  $V(G)$  and  $E(G)$  denote the vertex set and edge set of a graph  $G$ , and let  $\Gamma_G(v)$  denote the set of edges incident to a vertex  $v$ . An  $S$ -weighting of a graph  $G$  is a map  $w: E(G) \rightarrow S$ . A total  $S$ -weighting is a map  $w: E(G) \cup V(G) \rightarrow S$ . More specifically, a  $k$ -weighting is an  $S$ -weighting with  $S = \{1, \dots, k\}$ . For a weighting  $w$ , let  $\phi_w(v) = \sum_{e \in \Gamma_G(v)} w(e)$ . For a total weighting  $w$ , let  $\phi_w(v) = w(v) + \sum_{e \in \Gamma_G(v)} w(e)$ ; that is, each vertex is assigned the total of the weights it “sees”. A weighting or total weighting  $w$  is *proper* if  $\phi_w$  is a proper coloring of  $G$ . We seek proper  $k$ -weightings or proper total  $k$ -weightings for small  $k$ .

**Conjecture 1.1** (1, 2, 3-Conjecture; Karónski–Łuczak–Thomason [7]). *Every graph without isolated edges has a proper 3-weighting.*

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**Conjecture 1.2** (1, 2-Conjecture; Przybyło–Woźniak [8]). *Every graph has a proper total 2-weighting.*

Toward the 1, 2, 3-Conjecture, the original paper proved it for graphs with chromatic number at most 3. Addario-Berry et al. [1] showed that every graph without isolated edges has a proper  $k$ -weighting when  $k = 30$ . After improvements to  $k = 15$  in [2] and  $k = 13$  in [10], Kalkowski, Karóński, and Pfender [6] showed that every graph without isolated edges has a proper 5-weighting. Toward the 1, 2-Conjecture, Przybyło and Woźniak [8] proved it for complete graphs and for graphs with chromatic number at most 3. Kalkowski [5] showed that every graph has a proper total 3-weighting; furthermore, there is such a weighting with the edge weights in one spanning tree fixed arbitrarily and the vertex weights chosen from  $\{1, 2\}$ . Seamone [9] surveyed progress on these and related problems.

List versions of the conjectures have been proposed. A graph is  *$k$ -weight-choosable* if whenever each edge is given a list of  $k$  available integers, a proper weighting can be chosen from the lists. A graph is  *$(k, k')$ -weight-choosable* if whenever each vertex has a list of size  $k$  and each edge has a list of size  $k'$ , a proper total weighting can be chosen from the lists.

**Conjecture 1.3** (Bartnicki–Grytczuk–Niwczyk [4]). *Every graph without isolated edges is 3-weight-choosable.*

**Conjecture 1.4** (Wong–Zhu [12]). *Every graph is  $(2, 2)$ -weight-choosable. Every graph without isolated edges is  $(1, 3)$ -weight-choosable.*

These conjectures are stronger than the original conjectures, which consider only the special case of lists consisting of the smallest positive integers.

Wong, Yang, and Zhu [11] proved that the complete multipartite graph  $K_{n,m,1,1,\dots,1}$  is  $(2, 2)$ -weight-choosable and that complete bipartite graphs other than  $K_2$  are  $(1, 2)$ -weight-choosable. Bartnicki, Grytczuk, and Niwczyk [4] applied the Combinatorial Nullstellensatz [3] to prove Conjecture 1.3 for complete graphs, complete bipartite graphs, and trees. Wong and Zhu [12] applied the Combinatorial Nullstellensatz to prove Conjecture 1.4 for complements of linear forests; this includes complete graphs. They also proved that every tree with an even number of edges is  $(1, 2)$ -weight-choosable. Wong, Yang, and Zhu [11] continued this approach, proving Conjecture 1.4 for graphs with maximum degree 3. Finally, Wong and Zhu [13] proved that every graph is  $(2, 3)$ -weight-choosable.

Our main results show that various configurations cannot occur in minimal counterexamples to Conjectures 1.1 and 1.2. To discuss both problems together, let  *$j$ -weighting* mean a 3-weighting when  $j = 3$  and a total 2-weighting when  $j = 2$ . A graph is  *$j$ -bad* if it has no proper  $j$ -weighting (and no isolated edge if  $j = 3$ ). We say that a configuration that cannot occur in a minimal  $j$ -bad graph is  *$j$ -reducible*.

In this context, a *configuration* consists of a graph  $C$  plus labels  $f: V(C) \rightarrow \mathbb{N}$ . The configuration *occurs* in  $G$  if  $C$  appears as a subgraph  $H$  of  $G$  such that  $d_G(v) = f(v)$  for

$v \in V(H)$ . A configuration is proved  $j$ -reducible by showing that if it occurs in  $G$ , then  $G$  has an induced subgraph  $G'$  such that every proper  $j$ -weighting of  $G'$  yields a proper  $j$ -weighting of  $G$  (possibly by changing some weights before extending to  $G$ ).

Our reducibility results imply Conjectures 1.1 and 1.2 for sparse graphs via the “Discharging Method”. By “sparse”, we mean  $\text{Mad}(G) < c$  for some  $c$ , where the *maximum average degree*  $\text{Mad}(G)$  of a graph  $G$  is  $\max_{H \subseteq G} \bar{d}(H)$ , and  $\bar{d}(H) = \frac{\sum_{v \in V(H)} d_H(v)}{|V(H)|}$ . To show that  $\bar{d}(G) < c$  forces some desired configuration, we give each vertex initial charge equal to its degree and then shift charge so that if no specified configuration occurs, then every vertex has final charge at least  $c$ . Proving that  $\bar{d}(G) < 8/3$  forces a  $j$ -reducible configuration confirms the conjectures when  $\text{Mad}(G) < 8/3$ . However, that consequence is already known, since every subgraph of such a graph has a vertex with degree at most 2 and hence is 3-colorable. The original papers proved the conjectures for 3-colorable graphs.

The value of our results is thus separately in the structure theorems proved by discharging (which may be useful for other problems) and in the reducibility theorems prohibiting various configurations from minimal counterexamples. The reducibility proofs and discharging proofs are constructive and hence can be used in polynomial-time algorithms; the original proofs for 3-colorable graphs are also constructive.

Our proofs of  $j$ -reducibility use the restriction of weights to values at most  $j$ , so they do not extend to the list versions. Also, unlike in most coloring problems, vertices of degree 1 do not immediately yield reducible configurations, since the weight on a pendant edge affects whether its incident edges are properly colored.

For clarity, we first present in Section 2 short proofs that suffice when  $\text{Mad}(G) < 5/2$ . These arguments are used in and motivate the stronger results later. We begin with some 3-reducible configurations. We next show by discharging that  $\bar{d}(G) < 5/2$  forces some configuration in this set. We then prove 2-reducibility for similar configurations. These proofs of Conjectures 1.1 and 1.2 for  $\text{Mad}(G) < 5/2$  use the same discharging argument, although the sets of reducible configurations are different.

When  $\text{Mad}(G) \geq 5/2$ , no longer must  $G$  have a configuration among those in Section 2. In Section 3 and Section 4, respectively, we obtain sets of 2-reducible and 3-reducible configurations that are unavoidable when  $\text{Mad}(G) < 8/3$ .

## 2 Reducible Configurations and $\text{Mad}(G) < 5/2$

In discharging arguments for sparse graphs, it is convenient to have concise terminology for vertices satisfying degree constraints.

**Definition 2.1.** A vertex with degree  $k$ , at least  $k$ , or at most  $k$  is a  $k$ -vertex, a  $k^+$ -vertex, or a  $k^-$ -vertex, respectively. A  $j$ -neighbor of  $v$  is a  $j$ -vertex that is a neighbor of  $v$ .

Write  $N_G(v)$  for the neighborhood of  $v$  in  $G$  and  $d_G(v)$  for its degree. For  $v \in V(G)$  and  $U \subseteq N_G(v)$ , let  $[v, U]$  denote the set of edges joining  $v$  to  $U$ .

A weighting or total weighting  $w$  satisfies an edge  $uv$  if  $\phi_w(u) \neq \phi_w(v)$ , or equivalently if  $\rho_w(u, v) \neq \rho_w(v, u)$ , where we define  $\rho_w(x, y) = \phi_w(x) - w(xy)$  when  $x$  and  $y$  are adjacent.

A *configuration* in a graph  $G$  is a subgraph  $C$  together with specified degrees in  $G$  for  $V(C)$ . The *core* of the configuration is  $E(C)$ , and the resulting *derived graph* is  $G - E(C)$ .

We begin with a lemma that reduces the length of reducibility proofs: 1-neighbors are “easier” to handle than 2-neighbors, so when we claim that a configuration is reducible when a particular vertex has degree 1 or 2, in the proof we may assume that it has degree 2.

**Lemma 2.2.** *If a vertex  $z$  in a 2- or 3-reducible configuration  $C$  has degree 1 in  $C$  (with neighbor  $v$ ) and is specified as a 2-vertex of the full graph, then the configuration  $C'$  obtained from  $C$  by instead specifying degree 1 for  $z$  is also reducible.*

*Proof.* Let  $H$  be a graph containing  $C'$ , and let  $H'$  be the derived graph;  $z$  is isolated in  $H'$ . Form  $G$  by adding vertices  $a$  and  $b$  and edges  $ab$  and  $bz$  to  $H$ . Now  $C$  arises in  $G$ , and its derived graph  $G'$  arises from  $H'$  in the same way that  $G$  arises from  $H$ .

If  $H$  is a minimal  $j$ -bad graph, then  $H'$  has a desired weighting. Since also the path  $P_3$  has such a weighting,  $G'$  has such a weighting. Since  $C$  is  $j$ -reducible,  $G$  therefore also has such a weighting. To obtain the desired weighting of  $H$ , note that all edges are satisfied when  $a$  and  $b$  are deleted from the weighting of  $G$ , except possibly  $zv$ .

For  $j = 2$ , the weight on  $z$  is needed only to satisfy  $zv$  in  $H$  and can be respecified so that  $zv$  is satisfied. For  $j = 3$ , the edge  $zv$  is satisfied automatically since  $d_H(v) > 1$ .  $\square$

Reducibility proofs may use some types of inferences many times. The next lemma enables us to express statements concisely and reduce repetitive language. It can be stated in more generality, but for clarity we list just typical situations in which we will use it.

**Lemma 2.3.** *Let  $w$  be a partial  $j$ -weighting of a graph  $G$  ( $w$  is not specified everywhere). In the situations below, the weights on a set  $S$  can be chosen to satisfy the edges in a set  $F$  if the weights on all the edges (or vertices) incident to  $F$  and not in  $S$  are already known:*

- 1) *The edges of  $F$  have a common endpoint  $v$ , incident to all edges of  $S$  (possibly also  $v \in S$  when  $j = 2$ ), and  $|F| \leq (j - 1)|S|$ .*
- 2)  *$F$  consists of two edges,  $S$  is a single edge incident to both, and  $j = 3$ .*

*Proof.* Let  $k = |S|$ . Since weights are chosen from  $\{1, \dots, j\}$ , the sum of  $k$  weights has  $1 + (j - 1)k$  possible values. Each edge in  $F$  uses that sum in determining whether the values of  $\phi$  differ at its endpoints. Each edge in  $F$  thus forbids at most one value of the sum in a proper  $j$ -weighting. There are at least  $k(j - 1)$  possible augmentations above the least value of the sum, so when  $k(j - 1) \geq |F|$  the labels can be chosen to satisfy all of  $F$ .

Note that in (2) the weights on  $F$  may be unspecified; the weight on an edge does not affect whether it is satisfied. Similarly, if  $F = \{uv\}$ , and  $S$  is a single edge incident to  $v$  or is  $v$  itself, and the weights of all other items incident to  $uv$  are known, then the weight on  $S$  can be chosen in  $j - 1$  ways to satisfy  $F$ .  $\square$

**Remark 2.4.** We use Lemma 2.3 frequently in reducibility arguments, invoked without mention in 2-reducibility when we write “choose  $w(vz)$  to satisfy  $vx$ ” or “choose  $w(v)$  and  $w(vz)$  to satisfy  $vx$  and  $vv$ ”. In 3-reducibility, a choice can satisfy more. In 2-reducibility we can choose one weight to avoid one value, but in 3-reducibility it can avoid two values.

Another method of satisfying an edge  $uv$  is to create sufficient imbalance between the contributions at  $u$  and  $v$  to guarantee that  $\phi(u) \neq \phi(v)$  when the weighting is completed. When we write “Set  $w(uv) = 3$  to ensure satisfying  $vx$ ”, we mean that no way of choosing weights on the remaining edges can produce  $\phi(u) = \phi(v)$ . Saying that an edge is “automatically satisfied” has a similar meaning. For example, any edge joining a 1-vertex to a 3-vertex is automatically satisfied for (total) 2-weightings, while putting weight 1 at the 1-vertex ensures satisfying the edge even when the neighbor has degree 2.

The figures for configurations show the core in bold; the derived graph  $G'$  is obtained by deleting the core. Also, with  $w'$  assumed to be a proper  $j$ -weighting of  $G'$ , the label on an edge  $e$  is  $w'(e)$ , and the label in a circle at a vertex  $x$  with one neighbor  $u$  is  $\rho_{w'}(x, u)$ . To satisfy  $xu$ , the sum of the contributions at  $u$  other than  $w'(xu)$  must differ from  $\rho_{w'}(x, u)$ .

The figures do not show cases where some of the specified vertices may be equal. For instances where such equalities do not affect the validity of the written argument, we make no comment about possible changes in the illustration.

As mentioned earlier, proving reducibility for a configuration means modifying or extending a proper  $j$ -weighting  $w'$  of the derived graph  $G'$  to obtain a  $j$ -weighting  $w$  of  $G$  such that the edges in or incident to the core become satisfied, while the other edges of  $G'$  remain satisfied. If we do not change the weights on edges of  $G'$  incident to the core, then we do not change the fact that all edges of  $G'$  not incident to the core are satisfied.

With these preparations, we begin the reducibility arguments. The first lemma takes care of many degenerate cases of later configurations in which specified vertices may be identical.

**Lemma 2.5.** *The following configurations are both 2-reducible and 3-reducible:*

- (1) *A 3-cycle through two 2-vertices and one  $4^-$ -vertex.*
- (2) *A 3-cycle through one 2-vertex  $z$  and two vertices that each may be a 3-vertex, a 4-vertex with a 1-neighbor, or a 5-vertex with a 1-neighbor. In addition, one neighbor of  $z$  is allowed to be a 4-vertex with a 2-neighbor other than  $z$  (and no 1-neighbor).*

*Proof.* When  $G$  is a 3-cycle, the weights can be chosen to produce colors  $\{3, 4, 5\}$  at the vertices, for either value of  $j$ . When  $G \neq C_3$ , we extend a proper  $j$ -weighting  $w'$  of a subgraph  $G'$  obtained by deleting the core (see Figure 1).

For (1), let  $v$  be the  $3^+$ -vertex on the cycle, and let  $z$  and  $z'$  be the 2-vertices. To extend  $w'$  to  $w$ , first set  $w(zz') = 1$ , and also set  $w(v) = 2$  if  $j = 2$ . This ensures satisfying  $vz$  and  $vz'$ . If  $j = 2$ , then fix  $w(z) = 1$ , choose  $w(vz)$  and  $w(vz')$  to satisfy  $\Gamma_{G'}(v)$ , and choose  $w(z')$  to satisfy  $zz'$ . If  $j = 3$ , then require  $w(vz) \neq w(vz')$  with  $w(vz) \in \{1, 2\}$  and  $w(vz') \in \{2, 3\}$  to satisfy  $zz'$ . There are three choices for  $w(vz) + w(vz')$ , so we can choose them also to satisfy  $\Gamma_{G'}(v)$ , since  $d_{G'}(v) \leq 2$ .

For (2), let  $v$  and  $v'$  be the other vertices of the triangle. If  $d_G(v) \geq 4$ , then let  $u$  be a vertex of smallest degree in  $N(v) - \{z\}$ ; similarly define  $u' \in N(v')$ . Form  $G'$  from  $G$  by deleting  $\{vz, v'z, vv'\}$  and the edges  $vu$  and  $v'u'$  (if they exist). Figure 1 shows one of the possibilities at each of  $v$  and  $v'$ .

We first ensure that  $vz$  and  $v'z$  will be satisfied by setting  $w(vv') = j$  (and  $w(z) = 1$  if  $j = 2$ ); this will yield  $\rho_w(z, v) \leq 3 < 4 \leq \rho_w(v, z)$ , since  $d_G(v) \geq 3$ .

For  $d_G(v) = 3$ , choose  $w(vz)$  (and  $w(v)$  if  $j = 2$ ) to satisfy the one edge in  $\Gamma_{G'}(v)$ . For  $d_G(v) \in \{4, 5\}$  and  $d_G(u) = 1$ , choose  $w(vz)$  and  $w(vu)$  (and  $w(v)$  if  $j = 2$ ) to satisfy  $\Gamma_{G'}(v)$ . These cases have extra flexibility, so that if all contributions to  $\phi_w(v')$  are already known, then  $vv'$  can also be satisfied.

For  $d_G(v) = 4$  and  $d_G(u) = 2$ , choose  $w(vu)$  (and  $w(u)$  if  $j = 2$ ) to satisfy  $\Gamma_{G'}(u)$ , and then choose  $w(vz)$  (and  $w(v)$  if  $j = 2$ ) to satisfy  $vu$  and  $\Gamma_{G'}(v)$ . In this case we do not satisfy  $vv'$  using edges at  $v$ . Instead, we satisfy  $vv'$  using one of the earlier cases at  $v'$  after  $\phi_w(v)$  is known; this case is only allowed to occur at one of  $\{v, v'\}$ .  $\square$

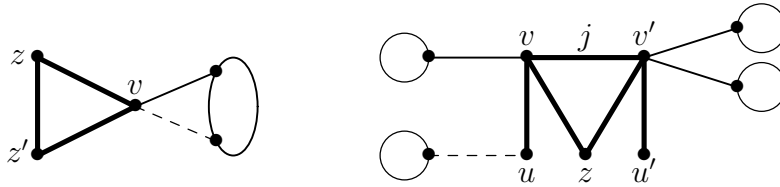


Figure 1: Cases (1) and (2) for Lemma 2.5

**Lemma 2.6.** *The following configurations are 3-reducible.*

- A. A 2-vertex or 3-vertex having a 1-neighbor.
- B. A  $4^-$ -vertex whose neighbors all have degree 2.
- C. A 3-vertex having two 2-neighbors, one of which has a 2-neighbor.
- D. A 4-vertex having a 1-neighbor and a  $2^-$ -neighbor.
- E. A  $5^+$ -vertex  $v$  with  $3p_1 + 2p_2 \geq d_G(v)$ , where  $p_i$  is the number of  $i$ -neighbors of  $v$ .

*Proof.* Let  $v$  be such a vertex in a graph  $G$ . Let  $U_i$  be the set of  $i$ -neighbors of  $v$ . Form  $G'$  as in Definition 2.1 (deleting the bold core), except that also any resulting isolated edges are deleted. We obtain a proper 3-weighting  $w$  of  $G$  from a proper 3-weighting  $w'$  of  $G'$ .

**Case A:**  $d_G(v) \leq 3$  and  $v$  has a 1-neighbor  $u$ . As in Lemma 2.3, we can choose  $w(uv)$  to satisfy the other edges at  $v$ . With  $d_G(v) \geq 2$ , the edge  $uv$  is automatically satisfied.

By Case A, deleting the core in Cases B,C,D leaves no isolated edges.

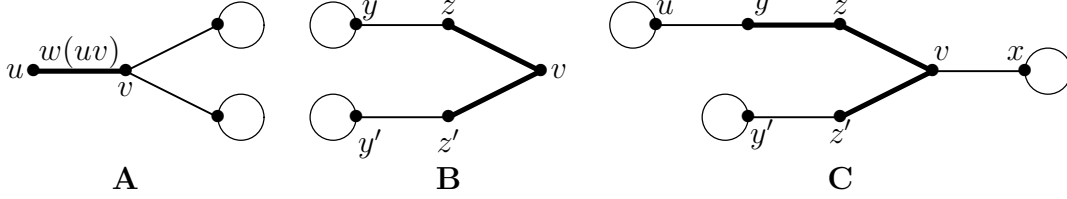


Figure 2: Cases A, B, C for Lemma 2.6

**Case B:**  $d_G(v) \leq 4$  and  $U_2 = N_G(v)$ . Let  $z$  and  $z'$  be 2-neighbors of  $v$ , with  $N_G(z) = \{v, y\}$  and  $N_G(z') = \{v, y'\}$ . By Lemma 2.5,  $\{y, y'\} \cap \{z, z'\} = \emptyset$ . Let  $G' = G - \{vz, vz'\}$ . For  $d_G(v) = 2$  (Figure 2B), choose  $w(vz)$  to satisfy  $yz$  and  $vz'$ , and  $w(vz')$  to satisfy  $y'z'$  and  $vz$ . If  $d_G(v) \in \{3, 4\}$ , then for  $z \in N_G(v)$  with  $zy \in E(G')$ , choose  $w(vz) \in \{2, 3\} - \{\rho_w(y, z)\}$  to satisfy  $yz$ . Since  $d(v) \geq 3 \geq w'(zy)$ , such choices on  $\Gamma_G(v)$  also satisfy  $zv$ .

**Case C:**  $d_G(v) = 3$  and  $U_2 = \{z, z'\}$ , with  $y$  a 2-neighbor of  $z$ . By Lemma 2.5,  $y \neq z'$ . Let  $G' = G - \{vz, vz', zy\}$ , leaving  $vx, z'y', yu \in E(G')$  (see Figure 2C). Choose  $w(vz)$  to satisfy  $zy$  and  $vz'$ , then  $w(vz')$  to satisfy  $z'y'$  and  $vx$ , and finally  $w(zy)$  to satisfy  $yu$  and  $vz$ .

**Case D:**  $d_G(v) = 4$  and  $N_G(v) = \{u, z, x, x'\}$  with  $d_G(u) = 1$  and  $d_G(z) \leq 2$ . By Lemma 2.2, we may assume  $d_G(z) = 2$ . Let  $G' = G - \{vu, vz\}$ , leaving  $zy \in E(G')$  (see Figure 3D, where  $y$  may be in  $N_G(v)$ ). When choosing  $w(vz)$  to satisfy  $zy$  and choosing  $w(vu)$  to satisfy  $vz$ , each has at least two possible values. Hence they can be chosen with three possible values for  $w(vz) + w(vu)$ , yielding a choice that also satisfies  $vx$  and  $vx'$ .

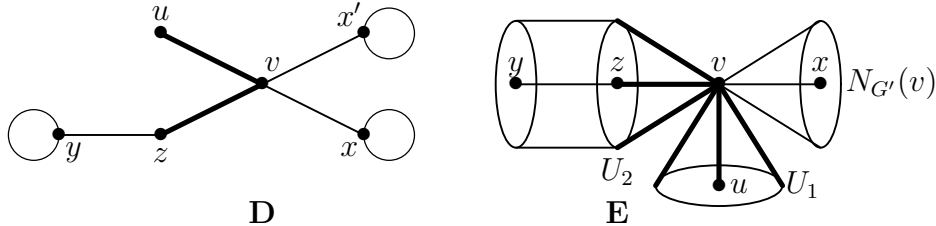


Figure 3: Cases D and E for Lemma 2.6

**Case E:**  $d_G(v) \geq 5$  and  $3p_1 + 2p_2 \geq d_G(v)$ . For  $z \in U_2$ , let  $y$  be the neighbor of  $z$  in  $G'$ . To satisfy  $yz$  when  $y \notin U_2$  (see Figure 3E), we need  $w(vz) \neq \rho_w(y, z)$ ; there are at least two such choices for  $w(vz)$ . (If  $y \in U_2$ , then  $yz$  is deleted in  $G'$ ; let  $w(yz) = 1$ . Now  $w(vy) \neq w(vz)$  is needed to satisfy  $yz$ , leaving three choices for  $w(vy) + w(vz)$ .)

Edges to  $U_1$  are automatically satisfied, since  $d_G(v) \geq 2$ . For  $z \in U_2$ , the edge  $zv$  will be satisfied, since  $d_G(v) \geq 5$  yields  $\rho_w(v, z) \geq 4 > 3 \geq w(zy)$ .

It remains to satisfy  $\Gamma_{G'}(v)$ . Let  $\sigma = \sum_{e \in E(G) - E(G')} w(e)$ . We need  $\sigma \neq \phi_{w'}(x) - \phi_{w'}(v)$  when  $x \in N_{G'}(v)$ , so  $\sigma$  must avoid  $d_G(v) - p_1 - p_2$  values. It suffices to have  $1 + 2p_1 + p_2$  choices for  $\sigma$ , since we are given  $2p_1 + p_2 \geq d_G(v) - p_1 - p_2$ . Weights on  $[v, U_1]$  have three choices. Weights on  $[v, U_2]$  have at least two choices, except that weights on two such edges incident to neighboring 2-vertices instead have three choices for their sum. Starting with the smallest choices, we can therefore make  $2p_1 + p_2$  augmentations to the sum using choices that satisfy the edges incident to  $U_2$ . Hence there are enough choices for  $\sigma$  to satisfy  $\Gamma_{G'}(v)$ .  $\square$

We now use discharging to obtain an unavoidable set of configurations.

**Lemma 2.7.** *Let  $G$  be a graph with no isolated edges. If  $\bar{d}(G) < 5/2$ , then  $G$  contains one of the following configurations.*

- A.** A 2-vertex or 3-vertex having a 1-neighbor.
- B.** A  $4^-$ -vertex whose neighbors all have degree 2.
- C.** A 3-vertex having two 2-neighbors, one of which has a 2-neighbor.
- D.** A 4-vertex having a 1-neighbor and a  $2^-$ -neighbor.
- E.** A  $5^+$ -vertex  $v$  with  $3p_1 + p_2 \geq 2d_G(v) - 4$ , where  $p_i$  is the number of  $i$ -neighbors of  $v$ .

*Proof.* We prove that avoiding **A-E** requires  $\bar{d}(G) \geq 5/2$ . Give each  $v \in V(G)$  initial charge  $d_G(v)$ . Move charge by the following rules:

- (1) Each  $4^+$ -vertex gives  $\frac{3}{2}$  to each 1-neighbor and  $\frac{1}{2}$  to each 2-neighbor.
- (2) Each 3-vertex with a 2-neighbor gives total  $\frac{1}{2}$  to its 2-neighbors, split equally if it has two 2-neighbors.

Let  $\mu(v)$  denote the resulting charge at  $v$ ; it suffices to check that  $\mu(v) \geq \frac{5}{2}$  for all  $v$ . For  $\mathbf{Z} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}\}$ , let  $\bar{\mathbf{Z}}$  mean ‘‘configuration  $\mathbf{Z}$  does not occur in  $G$ ’’.

Case  $d(v) = 1$ : By  $\bar{\mathbf{A}}$ , the neighbor of a 1-vertex  $v$  has degree at least 4, so  $\mu(v) = \frac{5}{2}$ .

Case  $d(v) = 2$ : By  $\bar{\mathbf{A}}$  and  $\bar{\mathbf{B}}$ ,  $v$  has a  $3^+$ -neighbor and receives from it  $\frac{1}{4}$  or  $\frac{1}{2}$ . If only  $\frac{1}{4}$ , then  $\bar{\mathbf{C}}$  implies that  $v$  also receives at least  $\frac{1}{4}$  from its other neighbor, so  $\mu(v) \geq \frac{5}{2}$ .

Case  $d(v) = 3$ : By  $\bar{\mathbf{A}}$  and  $\bar{\mathbf{B}}$ ,  $v$  has no 1-neighbor and at most two 2-neighbors. Hence  $v$  gives away 0 or  $\frac{1}{2}$ , and  $\mu(v) \geq \frac{5}{2}$ .

Case  $d(v) = 4$ : By  $\bar{\mathbf{D}}$  and  $\bar{\mathbf{B}}$ ,  $v$  has at most one 1-neighbor, has a 2-neighbor only if it has no 1-neighbor, and has at most three 2-neighbors. It loses at most  $\frac{3}{2}$ , and  $\mu(v) \geq \frac{5}{2}$ .

Case  $d(v) \geq 5$ :  $v$  gives  $\frac{3}{2}$  to each 1-neighbor and  $\frac{1}{2}$  to each 2-neighbor. By  $\bar{\mathbf{E}}$ ,  $\mu(v) = d_G(v) - \frac{1}{2}(3p_1 + p_2) \geq d_G(v) - \frac{1}{2}(2d_G(v) - 5) = \frac{5}{2}$ .  $\square$

**Theorem 2.8.** *If  $G$  has no isolated edge, and  $\text{Mad}(G) < \frac{5}{2}$ , then  $G$  has a proper 3-weighting.*

*Proof.* A minimal counterexample contains none of **A-E** in Lemma 2.6. Since it has average degree less than  $5/2$ , it contains a configuration in Lemma 2.7. The configurations are the same except for **E**. Since a  $5^+$ -vertex  $v$  satisfying  $3p_1 + p_2 \geq 2d_G(v) - 4$  also satisfies  $3p_1 + 2p_2 \geq d_G(v)$ , every graph with  $\text{Mad}(G) < 5/2$  contains a 3-reducible configuration.  $\square$



For the 1, 2-Conjecture, we again begin with reducible configurations. Isolated edges are now allowed, which eliminates some technicalities. We will use Lemma 2.7, but the list of 2-reducible configurations is different. In obtaining a proper total 2-weighting of  $G$  from such a weighting of a subgraph  $G'$ , we may erase weights from some vertices and recolor them.

Configuration **B** in the next lemma is more general than is needed for the 1, 2-Conjecture when  $\text{Mad}(G) < 5/2$ , but we will need its full generality in the proof for  $\text{Mad}(G) < 8/3$ .

**Lemma 2.9.** *A minimal 2-bad graph contains none of the following configurations.*

- A.** *A  $3^-$ -vertex having a 1-neighbor.*
- B.** *A  $4^-$ -vertex having two  $2^-$ -neighbors.*
- C.** *A  $5^+$ -vertex  $v$  whose number of  $2^-$ -neighbors is at least  $(d(v) - 1)/2$ .*

*Proof.* In each case, we obtain a proper total 2-weighting of  $G$  from such a weighting  $w'$  of the derived graph  $G'$ . Let  $v$  be the specified vertex. Since isolated edges have proper total 2-weightings, we may assume that any 1-vertex in  $G$  has a  $2^+$ -neighbor. In the extension arguments, we use Lemma 2.3 frequently to choose labels.

**Case A:**  $d(v) \leq 3$ , and  $v$  has a 1-neighbor  $u$ . For  $d(v) = 3$ , let  $N_{G'}(v) = \{x, x'\}$  (see Figure 4A). Uncolor  $v$ , and then choose  $w(v), w(uv) \in \{1, 2\}$  to satisfy  $vx$  and  $vx'$ . Now choose  $w(u)$  to satisfy  $uv$ . When  $d(v) = 2$ , we only need  $w(v) + w(uv)$  to avoid one value.

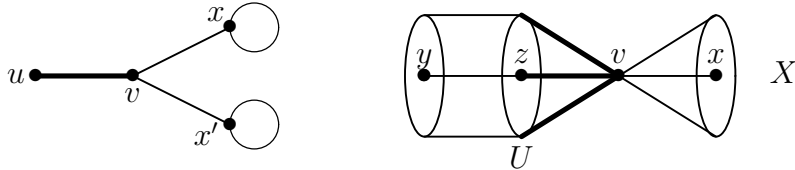


Figure 4: Cases **A** and **C** for Lemma 2.9

**Case C:**  $d(v) \geq 5$  and  $v$  has at least  $\frac{d(v)-1}{2}$   $2^-$ -neighbors. Let  $U$  be a set of  $p$  such neighbors, where  $p = \left\lceil \frac{d(v)-1}{2} \right\rceil$ , and let  $G' = G - [v, U]$  and  $X = N_{G'}(v)$ . By Lemma 2.2, for  $z \in U$  we may assume  $d_G(z) = 2$  and let  $\{y\} = N_{G'}(z)$  (see Figure 4C). Uncolor  $z$ .

By Lemma 2.3, we can choose the  $p+1$  weights on  $\{v\} \cup \{vz : z \in U\}$  to satisfy  $[v, X]$  in  $G$ , since  $p+1 \geq d(v) - p$ . Now choose  $w(z)$  for  $z \in U$  to satisfy  $zy$ . Finally, since  $d(v) \geq 5$ , we have  $\rho_w(v, z) \geq 5 > 4 \geq w(z) + w(zy)$ , so  $zv$  is automatically satisfied.

**Case B:**  $v$  has two  $2^-$ -neighbors  $z$  and  $z'$ . By Lemma 2.2, we may assume  $d_G(z) = d_G(z') = 2$ . By Lemma 2.5, we may assume  $zz' \notin E(G)$ . Let  $G' = G - \{vz, vz'\}$ . Since the edges of  $G'$  incident to the core must be satisfied in the extension to  $G$ , uncolor  $v$ ,  $z$ , and  $z'$ .

**Subcase 1:**  $d_G(v) \in \{3, 4\}$ . Let  $X = N_G(v) - \{z, z'\}$  (Figure 5). Let  $a = \sum_{x \in X} w'(vx)$ . If  $a \geq 2$ , then setting  $w(v) = 2$  ensures satisfying  $vz$  and  $vz'$ . Using  $|X| \leq 2$ , choose  $w(vz)$  and  $w(vz')$  to satisfy  $\Gamma_{G'}(v)$ . Now choose  $w(z)$  and  $w(z')$  to satisfy  $yz$  and  $y'z'$ , respectively.

Hence we may assume  $a = 1$ , which requires  $d_G(v) = 3$ . If  $w'(yz) = 1$ , then set  $w(vz') = 2$  to ensure satisfying  $vz$ . Next choose  $w(z')$  to satisfy  $z'y'$ , and then choose  $w(v)$  and  $w(vz)$  to satisfy  $vz'$  and  $vx$ . Finally, choose  $w(z)$  to satisfy  $yz$ .

By symmetry, we may now assume  $a = 1$  and  $w'(yz) = w'(y'z') = 2$ , as in the middle picture in Figure 5. If  $w'(y) = 1$ , then we can exchange  $w'(y)$  and  $w'(yz)$  with no effect on the satisfaction of any edge in  $\Gamma_{G'}(y)$  except  $yz$ , thereby reaching the case in the preceding paragraph. Hence by symmetry we may also assume  $w'(y) = w'(y') = 2$ .

Let  $b = \rho_{w'}(x, v)$ . If  $b = 4$ , then set  $w(zv) = w(v) = w(vz') = 2$  to satisfy  $vx$  and ensure satisfying  $vz$  and  $vz'$ ; then choose  $w(z)$  and  $w(z')$  to satisfy  $zy$  and  $z'y'$ , respectively. If  $b \neq 4$ , then set  $w(v) = 2$  and  $w(z) = w(zv) = w(vz') = w(z') = 1$ . By  $\overline{\mathbf{A}}$ , we have  $d_G(y) \geq 2$  and hence  $\rho_{w'}(y, z) > 2$ , so  $yz$  is satisfied (similarly for  $y'z'$ ). These values also satisfy  $\Gamma_G(v)$ .

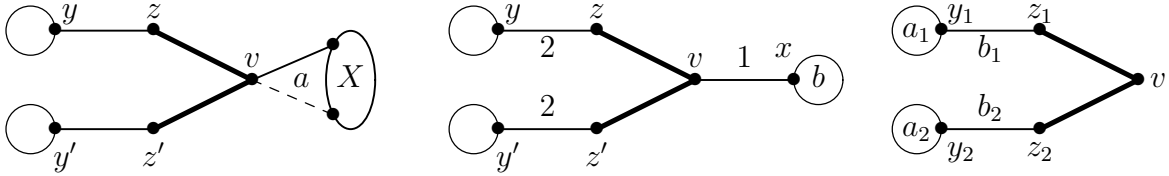


Figure 5: Case **B** for Lemma 3.4

**Subcase 2:**  $d_G(v) = 2$ . For this subcase, let  $z_1 = z$  and  $z_2 = z'$ . For  $i \in \{1, 2\}$ , let  $\{y_i\} = N_{G'}(z_i)$ , let  $b_i = w'(y_i z_i)$ , and let  $a_i = \rho_{w'}(y_i, z_i)$ , as on the right in Figure 2.9. To satisfy  $y_i z_i$ , fix  $w(vz_i) = 3 - w(z_i)$  when  $a_i$  is even and  $w(vz_i) = w(z_i)$  when  $a_i$  is odd. We then must choose  $w(z_1)$  and  $w(z_2)$  (and hence  $w(vz_1)$  and  $w(vz_2)$ ) to satisfy  $vz_2$  and  $vz_1$ . We need  $b_1 + w(z_1) \neq w(v) + w(vz_2)$  and  $b_2 + w(z_2) \neq w(v) + w(vz_1)$ .

When  $a_1 - a_2$  is even, set  $w(v) = 1$ . When  $a_1$  and  $a_2$  are both even, using  $w(vz_i) = 3 - w(z_i)$  converts the requirements to  $b_1 + w(z_1) \neq 4 - w(z_2)$  and  $b_2 + w(z_2) \neq 4 - w(z_1)$ . With two choices for both  $w(z_1)$  and  $w(z_2)$ , we can pick them so that  $w(z_1) + w(z_2) \notin \{4 - b_1, 4 - b_2\}$ . When  $a_1$  and  $a_2$  are both odd, using  $w(vz_i) = w(z_i)$  it suffices to choose  $w(z_1), w(z_2) \in \{1, 2\}$  so that  $w(z_1) - w(z_2) \notin \{1 - b_1, b_2 - 1\}$ . Since the difference can be any of the three values in  $\{1, 0, -1\}$ , this also can be done.

When  $a_1$  and  $a_2$  have opposite parity, we may assume that  $a_1$  is even. Now set  $w(v) = 3 - b_1$  and  $w(z_1) = w(z_2) = b_1$ . Using  $w(vz_1) = 3 - w(z_1)$  and  $w(vz_2) = w(z_2)$ , we have satisfied  $vz_1$  because  $b_1 + w(z_1) = 2b_1 \neq 3 = w(v) + w(z_2)$ , and we have satisfied  $vz_2$  because  $b_2 + w(z_2) = b_2 + b_1 \neq 6 - 2b_1 = w(v) + 3 - w(z_1)$ .  $\square$

**Theorem 2.10.** *If  $\text{Mad}(G) < 5/2$ , then  $G$  has a proper total 2-weighting.*

*Proof.* By Lemma 2.9, a minimal counterexample contains no configuration listed there. Since it has average degree less than  $5/2$ , it contains a configuration listed in Lemma 2.7. Configurations **A–D** are all 2-reducible, by **A** and **B** of Lemma 2.9. Hence to show that

every graph with  $\text{Mad}(G) < 5/2$  contains a 2-reducible configuration, it suffices to show that a  $5^+$ -vertex  $v$  satisfying  $3p_1 + p_2 \geq 2d_G(v) - 4$  also satisfies  $2p_1 + 2p_2 \geq d_G(v) - 1$ . If the desired inequality fails, then subtracting  $2p_1 + 2p_2 \leq d_G(v) - 2$  from the given inequality yields  $p_1 \geq d_G(v) - 2$ . Since  $d_G(v) - 2 \geq (d_G(v) - 1)/2$ , the desired inequality follows.  $\square$

### 3 Proper Total 2-weighting when $\text{Mad}(G) < 8/3$

A graph formed by adding a pendant edge at each vertex of a 3-regular graph has average degree  $5/2$ . It has no configuration in Lemma 2.9, since each 4-vertex has one 1-neighbor and three 4-neighbors. Further 2-reducible configurations will require multiple “almost-reducible” vertices. We introduce two types.

**Definition 3.1.** A  $\beta$ -vertex is a 3-vertex having exactly one 2-neighbor and no 1-neighbor. A  $\beta'$ -vertex is a  $2k$ -vertex, where  $k \geq 2$ , having exactly  $k - 1$  neighbors of degree 1 and no 2-neighbor. For  $\gamma \in \{\beta, \beta'\}$ , a  $\gamma$ -neighbor of  $v$  is a  $\gamma$ -vertex in  $N(v)$ .

We will show in Lemma 3.4 that various configurations involving such vertices are 2-reducible. Theorem 3.5 shows that these plus the configurations in Lemma 2.9 form an unavoidable set when  $\text{Mad}(G) < 8/3$ . The argument would be shorter if adjacent  $\beta'$ -vertices of degree 4 formed a reducible configuration, but our usual method fails there.

**Example 3.2.** Let  $v$  and  $v'$  be adjacent  $\beta'$ -vertices of degree 4 in  $G$ , having 1-neighbors  $u$  and  $u'$ , respectively. As in Section 2, the core  $F$  is  $\{uv, vv', v'u'\}$ , and  $G' = G - F$ . A total 2-weighting  $w'$  of  $G'$  may assign labels as indicated in Figure 6. To extend  $w'$ , we need  $w(uv) + w(v) + w(vv') \in \{3, 6\}$ ; hence these three weights must be equal. Similarly,  $w(u'v') + w(v') + w(vv') \in \{3, 6\}$ . Since  $w(vv')$  can take only one value, we have forced  $\phi_w(v) = \phi_w(v')$ . Hence no extension to a proper total 2-weighting is possible.

Another would-be-useful but non-reducible configuration consists of a  $\beta$ -vertex  $v$  whose 2-neighbor  $z$  has a 2-neighbor  $y$ . A total 2-weighting  $w'$  of  $G'$  may assign labels as indicated in Figure 6. The values of  $\phi_w'$  at the neighbors of  $v$  other than  $z$  force  $w(v) = w(vz) = 1$ . Now satisfying  $vz$  requires  $w(z) = w(zv)$ . Similarly, satisfying  $xy$  requires  $w(y) = w(yz)$ . We conclude  $w(z) = w(y)$ , but now  $yz$  cannot be satisfied, since also  $w(yx) = w(zv)$ .  $\square$

Graphs formed by adding a pendant edge at each vertex of a 3-regular graph contain only configuration **F** among those in Lemma 3.4. Although its reducibility proof does not require the full flexibility of choosing weights in it, Example 3.2 shows that the local argument cannot be completed when a  $\beta'$ -vertex has only one  $\beta'$ -neighbor (of degree 4).

Example 3.2 also shows that a  $\beta$ -vertex is not reducible, even when its 2-neighbor has another 2-neighbor. Nevertheless, when a  $\beta$ -vertex appears in a minimal 2-bad graph we can guarantee satisfying all but one specified edge at that vertex. This is useful when we can ensure satisfying that edge, such as when its other endpoint has high degree.

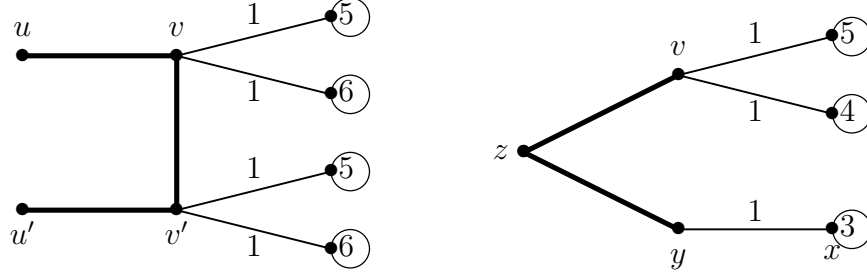


Figure 6: Non-reducible: adjacent  $\beta'$ -vertices, or  $\beta$ -vertex near extra 2-vertex

**Lemma 3.3.** *Let  $v$  be a  $\beta$ -vertex with 2-neighbor  $z$  in a minimal 2-bad graph  $G$ . Let  $N_G(v) = \{z, x, u\}$  and  $N_G(z) = \{v, y\}$  (see Figure 7). If  $G - vz$  has a proper partial 2-weighting  $w'$  satisfying  $\Gamma_G(x)$  and  $\Gamma_G(y)$ , then  $G$  has a partial 2-weighting  $w$  satisfying the same edges other than  $vu$ , plus  $vz$ , without changing weights on  $G - \{v, z\}$  except possibly on  $yz$  and  $y$ .*

*Proof.* Let  $G' = G - vz$ . By Lemma 2.9A,  $d(y) \geq 2$ . We want to choose  $w(v)$ ,  $w(z)$ , and  $w(vz)$  to satisfy  $\{xv, vz, zy\}$ , leaving edges other than  $vu$  satisfied.

Let  $a = \rho_{w'}(y, z)$ . If  $a \geq 4$ , then setting  $w(zv) = 1$  ensures satisfying  $yz$ , after which we choose  $w(v)$  to satisfy  $vx$  and  $w(z)$  to satisfy  $vz$ .

If  $a = 3$  and  $d_G(y) = 3$ , then  $w'(y) = 1$  ( $y = u$  is allowed). If  $w'(yz) = 2$ , then we can exchange the weights on  $y$  and  $yz$  and apply the previous case. If  $w'(yz) = 1$ , then setting  $w(z) = w(zv) = 1$  satisfies both  $yz$  and  $zv$ , after which we choose  $w(v)$  to satisfy  $vx$ .

The remaining case is  $d_G(y) = 2$ ; let  $N_G(y) = \{z, u'\}$  ( $u' = u$  is allowed). Uncolor  $y$  and  $yz$ . Setting  $w(yz) = 1$  and  $w(v) = 2$  ensures satisfying  $zv$ . Now choose  $w(vz)$  to satisfy  $vx$ ,  $w(y)$  to satisfy  $yu'$ , and  $w(z)$  to satisfy  $yz$ .  $\square$

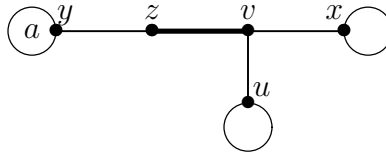


Figure 7: Configuration for Lemma 3.3

**Lemma 3.4.** *The configurations below are 2-reducible.*

- A. A  $3^-$ -vertex having a 1-neighbor.
- B. A  $4^-$ -vertex having two  $2^-$ -neighbors.
- C. A  $5^+$ -vertex  $v$  whose number of  $2^-$ -neighbors is at least  $(d_G(v) - 1)/2$ .
- D. Two adjacent  $\beta$ -vertices.
- E. A  $\beta$ -vertex with a  $\beta'$ -neighbor.
- F. A  $\beta'$ -vertex of degree 4 having two  $\beta'$ -neighbors of degree 4.
- G. A 3-vertex such that each neighbor is a  $\beta$ -vertex or is a  $\beta'$ -vertex of degree 4.

*Proof.* Configurations **A–C** were shown to be 2-reducible in Lemma 2.9. For **D–G**, as usual we consider a minimal 2-bad graph  $G$  containing the specified configuration, and the derived graph  $G'$  is obtained by deleting the core, shown in bold in Figures 8–10. In each case we have a proper total 2-weighting  $w'$  of  $G'$  and produce a proper total 2-weighting  $w$  of  $G$  by choosing weights on the deleted edges and on their endpoints, leaving all other weights as in  $w'$ , with the possible exception of applying Lemma 3.3. For each successive configuration, we know that the earlier configurations do not occur in  $G$ .

**Case D:**  $v$  and  $v'$  are adjacent  $\beta$ -vertices. As shown in Figure 8D,  $v$  and  $v'$  have degree 3, with 2-neighbors  $z$  and  $z'$ , respectively. Let  $N_G(v) = \{z, v', x\}$  and  $N_G(v') = \{z', v, x'\}$ , also  $N_G(z) = \{v, y\}$  and  $N_G(z') = \{v, y'\}$  ( $y = y'$  and/or  $x = x'$  are allowed in the argument). By Lemma 2.5, we may assume  $z \neq z'$ ,  $y \neq x$ , and  $y' \neq x'$ . Let  $G' = G - \{zv, vv', v'z'\}$ .

Consider first the degenerate case  $zz' \in E(G)$ , so  $y = z'$  and  $y' = z$ . This also handles the case  $y = x'$  or  $y' = x$  under appropriate relabeling. Set  $w(zz') = 1$  and  $w(vv') = 2$  to ensure satisfying  $zv$  and  $z'v'$ . Set  $w(z) = w(zv) = 2$ . Now choose  $w(v)$  to satisfy  $zv$ , choose  $w(v')$  and  $w(v'z')$  to satisfy  $vv'$  and  $v'x'$ , and choose  $w(z')$  to satisfy  $zz'$ .

Hence we may assume that the vertices are distinct as on the left in Figure 8D. Let  $a = w'(zy)$ ,  $b = \rho_{w'}(y, z)$ ,  $c = w'(vx)$ , and  $d = \rho_{w'}(x, v)$ . Define  $a', b', c', d'$  analogously using  $y', z', v', x'$ . In all subcases, set  $w(vv') = 2$ .

**Subcase 1:**  $d_G(y) = 2$  (or  $d_G(y') = 2$ ). Let  $N_G(y) = \{z, u\}$ . Uncolor  $u$  and  $yu$ . Treat  $y = z'$  (which implies  $u = v'$ ) as a special case. When  $y = z'$ , set  $w(zz') = w(z') = 1$ ; in general, set  $w(z) = w(z') = 1$ . In both cases, this ensures satisfying  $vz$  and  $v'z'$  (since  $w(vv') = 2$ ). Now set  $w(v'z') = 1$  when  $y = z'$ ; otherwise choose  $w(v'z')$  to satisfy  $y'z'$ . In both cases, next choose  $w(v')$  to satisfy  $v'x'$ , then  $w(v)$  and  $w(vz)$  to satisfy  $vv'$  and  $vx$ . Finally, choose  $w(z)$  to satisfy  $zz'$  when  $y = z'$ ; otherwise, choose  $w(u)$  to satisfy  $yu$  and then  $w(yu)$  to satisfy the other edge at  $u$ .

**Subcase 2:**  $d_G(y) \neq 2$ . By  $\bar{A}$ , we may assume  $d(y) \geq 3$ . If  $c = 2$  or  $a = 1$ , then  $w(vv') = 2$  ensures satisfying  $zv$ . Set  $w(v') = 2$  to guarantee satisfying  $z'v'$ . Now choose  $w(z'v')$  to satisfy  $v'x'$ , and choose  $w(z')$  to satisfy  $z'y'$ . Next choose  $w(zv)$  and  $w(v)$  to satisfy  $vx$  and  $vv'$ . Finally, choose  $w(z)$  to satisfy  $zy$ .

We may therefore assume  $c = 1$  and  $a = 2$ , and by symmetry  $c' = 1$  and  $a' = 2$ . We may also assume  $w'(y) = w'(y') = 2$ , since otherwise we can switch weights on  $y$  and  $yz$  (or on  $y'$  and  $y'z'$ ), which leaves the other edges at  $y$  or  $y'$  satisfied and yields the subcase above.

With  $d_G(y) \geq 3$ , we have  $b \geq 4$  (since  $w'(y) = 2$ ). By symmetry,  $d_G(y') \geq 3$  and  $b' \geq 4$ . Now setting  $w(z) = w(z') = 1$  ensures satisfying  $\Gamma_G(z)$  and  $\Gamma_G(z')$  (since  $w(vv') = 2$ ). Finally, choose  $w(zv) + w(v)$  to avoid  $d - 2$  and  $w(z'v') + w(v')$  to avoid  $d' - 2$  (allowing two choices for each sum) so that the sums are different. This satisfies  $vx$ ,  $v'x'$ , and  $vv'$ .

**Case E:**  $v$  is a  $\beta$ -vertex with a  $\beta'$ -neighbor  $v'$ . Let  $N_G(v) = \{x, z, v'\}$  with  $N_G(z) = \{y, v\}$ . If  $d_G(v') \geq 6$ , then let  $U$  be the set of 1-neighbors of  $v'$ . Set  $w(v') = 2$  to guarantee satisfying

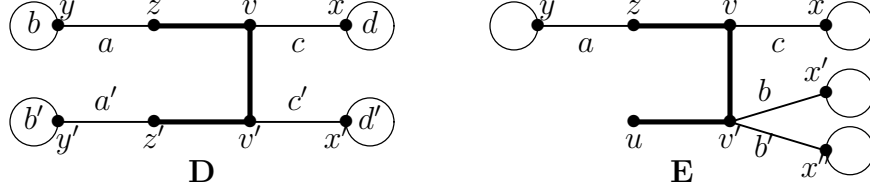


Figure 8: Cases **D** and **E** for Lemma 3.4

$vv'$  (since now  $\rho_w(v', v) \geq 7 > 6 \geq \rho_w(v, v')$ ). Including  $vv'$ , there remain  $|[v', N_G(v') - U]|$  edges incident to  $v'$  with unchosen weights; by Lemma 2.3, we can choose them to satisfy  $[v', N_G(v') - U]$ . The edges of  $[v, U]$  are automatically satisfied. Finally, with  $w(vv')$  chosen, Lemma 3.3 allows us to complete  $w$  to a proper 2-weighting of  $G$ .

We may therefore assume  $d_G(v') = 4$ , as in Figure 8E. Let  $u$  be the 1-neighbor of  $v'$ . Let  $G' = G - \{zv, vv', v'u\}$ , leaving  $v'x', v'x'' \in E(G')$ . Let  $a = w'(zy)$ ,  $b = w'(v'x')$ ,  $b' = w'(v'x'')$ , and  $c = w'(vx)$ . The argument allows  $y \in \{x', x''\}$ . Fix  $w(u) = 1$ .

If  $b + b' - c \geq 2$ , then requiring  $w(v) + w(zv) = 3$  guarantees satisfying  $v'v$ . Next choose  $w(v'v)$  to satisfy  $vx$ , and choose  $w(v')$  and  $w(v'u)$  to satisfy  $v'x'$  and  $v'x''$ . With  $w(v) = 3 - w(zv)$ , there are three choices for  $w(zv) + w(z)$ , so we can choose  $w(zv)$  and  $w(z)$  with  $w(z) + w(zv) \neq \rho_{w'}(y, z)$  to satisfy  $yz$  and  $w(z) + a \neq 3 - w(zv) + c + w(vv')$  to satisfy  $zv$ . We may therefore assume  $b + b' - c \leq 1$ .

If  $c = 2$  or  $a = 1$ , then requiring  $w(v) + w(vv') = 3$  guarantees satisfying  $zv$ . Now choose  $w(zv)$  to satisfy  $vx$  and then  $w(z)$  to satisfy  $zy$ . Finally, tentatively set  $w(vv') = 2$  and  $w(v) = 1$ , and then choose  $w(v')$  and  $w(v'u)$  to satisfy  $v'x'$  and  $v'x''$ . If  $vv'$  is not now satisfied, then  $w(v') + w(v'u) < 4$ . Now exchange weights on  $vv'$  and  $v$  while increasing  $w(v')$  or  $w(v'u)$  to satisfy  $vv'$  and preserve the satisfaction of  $v'x'$  and  $v'x''$ .

Hence we may assume  $c = 1$  and  $a = 2$ . Since also  $b + b' - c \leq 1$ , we have  $b = b' = 1$ . Tentatively set  $w(v'u) = 2$ , and choose  $w(v')$  and  $w(vv')$  to satisfy  $v'x'$  and  $v'x''$ , with  $w(v') \geq w(vv')$ . If  $w(v') = 2$ , then  $vv'$  is automatically satisfied (since  $c = 1$ ), and Lemma 3.3 completes the extension to  $w$ . If  $w(v') = 1$  and the application of Lemma 3.3 produces  $w(zv) = w(v) = 2$ , then  $vv'$  is not satisfied. In this case,  $\rho_{w'}(x', v'), \rho_{w'}(x'', v') = \{6, 7\}$ , and changing  $w(v'u)$  to 1 satisfies  $vv'$  while still satisfying  $v'x'$  and  $v'x''$ .

**Case F:**  $v$  is a  $\beta'$ -vertex of degree 4 with  $\beta'$ -neighbors  $z$  and  $z'$  of degree 4. Let  $N_G(v) = \{x, u, z, z'\}$ . Let  $u, y, y'$  be the 1-neighbors of  $v, z, z'$ , respectively. (see Figure 9).

**Subcase 1:**  $zz' \in E(G)$ . Here we have a triangle of  $\beta'$ -vertices with degree 4. Let  $G' = G - \{vz, vz', zz', vu, zy, z'y'\}$ . Let  $t$  and  $t'$  be the remaining neighbors of  $z$  and  $z'$ , respectively. By symmetry, the only cases are  $w'(zt) \neq w'(z't')$  or  $w'(zt) = w'(z't') = w'(vx) = c$ .

If  $w'(zt) \neq w'(z't')$ , then by symmetry we may assume  $w'(zt) = 1$  and  $w'(z't') = 2$ . Set  $w(zz') = w(z') = w(z'v) = 2$  and  $w(zv) = w(v) = 1$  to ensure satisfying  $zz'$  and  $z'v$ . Now choose  $w(vu)$  to satisfy  $\Gamma_{G'}(v)$  and choose  $w(z'y')$  to satisfy  $\Gamma_{G'}(z')$ . Finally, choose  $w(z)$

and  $w(zy)$  to satisfy  $zv$  and  $\Gamma_{G'}(z)$ .

In the other case, let  $w(vu) = w(v) = w(vz) = w(vz') = a$ . Choose  $a$  to satisfy  $vx$ . Let  $w(zz') = 3 - a$ ; this ensures satisfying  $vz$  and  $vz'$ . With  $w(z')$  arbitrary, choose  $w(z'y')$  to satisfy  $z't'$ , and finally choose  $w(z)$  and  $w(zy)$  to satisfy  $zz'$  and  $zt$ .

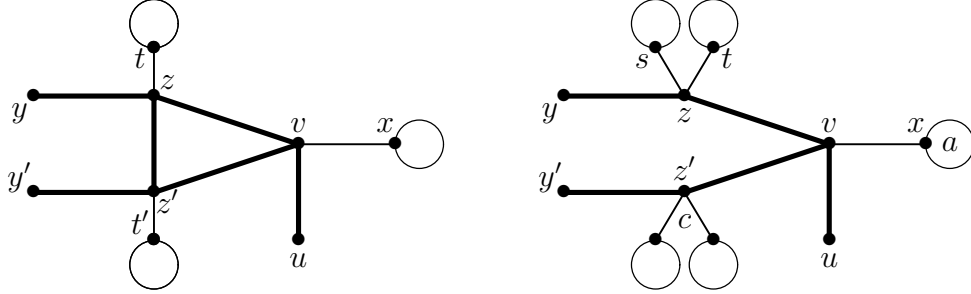


Figure 9: Cases **F1** and **F2** for Lemma 3.4

**Subcase 2:**  $zz' \notin E(G)$ . Let  $G' = G - \{yz, zv, vu, vz', z'y'\}$ . Let  $a = \rho_{w'}(x, v)$ . If  $a \neq 6$ , then requiring  $w(v) + w(vz') = 3$  and  $w(vz) + w(vu) = 3$  satisfies  $xv$ , with  $w(vz')$  and  $w(vz)$  still choosable freely. Choose  $w(zy)$ ,  $w(z)$ , and  $w(vz)$  so that their sum avoids  $\{\rho_{w'}(s, z), \rho_{w'}(t, z)\}$ , where  $\{s, t\} = N_{G'}(z)$ , and so that  $w(zy) + w(z) + w'(sz) + w'(tz) \neq 3 - w(vz) + w(v) + w(vz') + w'(vx)$  (to satisfy  $vz$ ). Since  $w(v) + w(vz') = 3$ , there are three constants for  $w(zy) + w(z) + w(vz)$  to avoid, so such a choice exists. Finally, choose  $w(vz')$ ,  $w(z')$ , and  $w(z'y')$  to satisfy  $vz$  and  $\Gamma_{G'}(z)$ , again making their sum avoid three known values.

Hence we may assume  $a = 6$ . Now choosing  $w(v) = w(vu) = w(vz)$  guarantees satisfying  $vx$ ; let  $b$  denote the value to be chosen for them. Let  $c$  be the total weight assigned by  $w'$  to  $\Gamma_{G'}(z')$ . If  $c = 2$ , or if  $c = 3$  and  $w(vx) = 2$ , then let  $b = 2$ . Otherwise, let  $b = 1$ . In either case,  $vz'$  is guaranteed to be satisfied. Finally, choose  $w(z)$  and  $w(zy)$  to satisfy  $zs$  and  $zt$ , choose  $w(vz')$  to satisfy  $vz$ , and choose  $w(z')$  and  $w(z'y')$  to satisfy  $\Gamma_{G'}(z')$ .

**Case G:**  $v$  is a 3-vertex with neighbors  $z_1, z_2, z_3$  (where  $d_G(z_1) \geq d_G(z_2) \geq d_G(z_3)$ ) such that each is a  $\beta$ -vertex or is a  $\beta'$ -vertex of degree 4. For  $i \in \{1, 2, 3\}$ , let  $y_i$  be the neighbor of  $z_i$  with degree  $5 - d_G(z_i)$ . When  $d_G(z_i) = 3$ , let  $x_i$  be the other neighbor of  $z_i$  and let  $y'_i$  be the other neighbor of  $y_i$ . When  $d_G(z_i) = 4$ , let  $\{x_i, x'_i\} = N_G(z_i) - \{v, y_i\}$ .

We first reduce to the case where  $N_G(v)$  is independent. Adjacent  $\beta$ -vertices are forbidden by **D**. Adjacent  $\beta$ - and  $\beta'$ -vertices are forbidden by **E**.

The third possibility is that  $z$  and  $z'$  are adjacent  $\beta'$ -vertices having a common 3-neighbor  $v$ . The situation is illustrated by deleting  $u$  from the left graph in Figure 9. Label the vertices as described there, with  $G' = G - \{vz, vz', zz', zy, z'y'\}$ . If  $w'(zt) = 2$  or  $w'(vx) = 1$ , then set  $w(vz) = w(vz') = 1$  and  $w(z') = w(zz') = 2$  to ensure satisfying  $vz$  and  $vz'$ . Choose  $w(v)$  to satisfy  $vx$ , choose  $w(z'y')$  to satisfy  $z't'$ , and choose  $w(z)$  and  $w(zy)$  to satisfy  $zt$  and  $zz'$ .

By symmetry, we may now assume  $w'(zt) = w'(z't') = 1$  and  $w'(vx) = 2$ . Also,  $w'(x) = 2$ , or we can switch the weights on  $x$  and  $vx$  to reach the case just discussed. Now, using  $d(x) \geq 3$  (since  $x$  is a  $\beta$ - or  $\beta'$ -neighbor of  $v$ ), we have  $\rho_{w'}(x, v) \geq 4$ . Now set  $w(zv) = w(v) = w(vz') = 1$  and  $w(zz') = 2$  to satisfy  $vx$  and ensure satisfying  $vz$  and  $vz'$ . Finally, set  $w(z') = 1$ , choose  $w(z'y')$  to satisfy  $z't'$ , and choose  $w(z)$  and  $w(zy)$  to satisfy  $zt$  and  $zz'$ .

Hence we may assume that  $N_G(v)$  is independent. If two  $\beta$ -vertices in  $N_G(v)$  have a common 2-neighbor, say  $z_1$  and  $z_2$  with common 2-neighbor  $y$ , then let  $G' = G - \{vz_1, vz_2, z_1y, z_2y\}$ . Set  $w(vz_1) = w(vz_2) = 2$  and  $w(y) = 1$  to ensure satisfying  $z_1y$  and  $z_2y$ . Now choose  $w(v)$  to satisfy  $vz_3$ , choose  $w(z_1)$  and  $w(z_1y)$  to satisfy  $vz_1$  and  $\Gamma_{G'}(z_1)$ , and choose  $w(z_2)$  and  $w(z_2y)$  to satisfy  $vz_2$  and  $\Gamma_{G'}(z_2)$ .

Now  $N_G(v)$  independent and the 2-neighbors of  $\beta$ -neighbors of  $v$  are distinct. The remaining cases are shown in Figure 10. The argument does not require the vertices on circles to be distinct. Let Subcase  $j$  be the situation where  $j$  neighbors of  $v$  are  $\beta'$ -vertices. In each subcase, the deleted core consists of  $\Gamma_G(v)$  and  $\{z_1y_1, z_2y_2, z_3y_3\}$ . To obtain  $w$  from  $w'$ , we must satisfy these six edges and six additional edges incident to them. We have the freedom to choose weights on the six deleted edges and their seven incident vertices.

We define operation  $S_i$  to satisfy the edges in the  $i$ th “branch” when  $w(vz_i)$  has been specified. If  $d_G(z_i) = 3$ , then  $S_i$  uses Lemma 3.3 to choose  $w(z_i)$ ,  $w(z_iy_i)$ , and  $w(y_i)$  (plus possible changes to weights on  $y_iy'_i$  and  $y'_i$  but not on  $z_ix_i$  or  $z_iv$ ) so that  $z_ix_i$ ,  $z_iy_i$ , and  $y_iy'_i$  become satisfied. If  $d_G(z_i) = 4$ , then  $S_i$  chooses  $w(z_i)$  and  $w(z_iy_i)$  to satisfy  $z_ix_i$  and  $z_ix'_i$ . (When  $d_G(z_i) = 4$ , automatically  $z_iy_i$  is satisfied, and  $w(y_i)$  is irrelevant.)

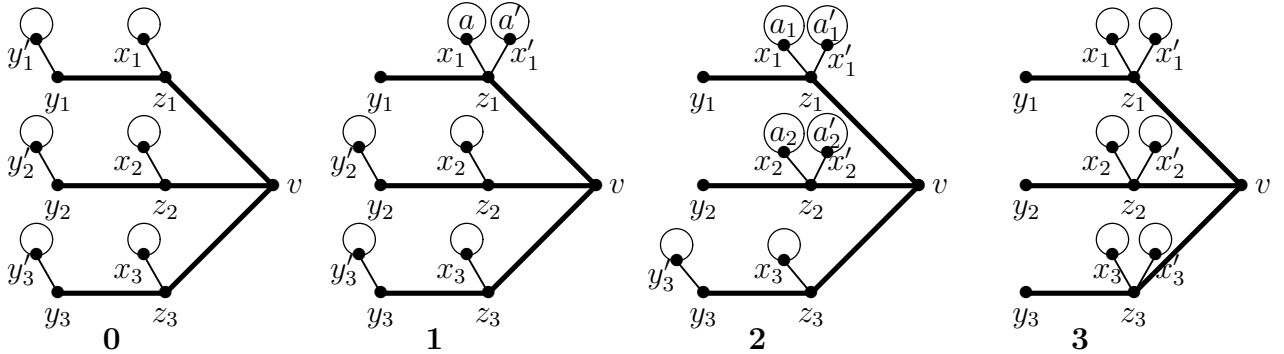


Figure 10: Case **G** for Lemma 3.4

**Subcase 0:** Set  $w(v) = w(vz_1) = w(vz_2) = w(vz_3) = 2$ , and consider  $i \in \{1, 2, 3\}$ . If  $w'(x_iz_i) = 1$ , then  $z_iv$  is automatically satisfied; apply  $S_i$ . If  $w'(x_iz_i) = 2$ , then  $z_iy_i$  is automatically satisfied. Set  $w(z_i) = 1$  to satisfy  $z_iv$ . Choose  $w(z_iy_i)$  to satisfy  $z_ix_i$ , and choose  $w(y_i)$  to satisfy  $y_iy'_i$ .

**Subcase 1:** Let  $a = \rho_{w'}(x_1, z_1)$  and  $a' = \rho_{w'}(x'_1, z_1)$ . When  $w'(z_1x_1) = w'(z_1x'_1) = 1$  and  $\{a, a'\} = \{6, 7\}$ , set  $w(z_3v) = w(z_2v) = 2$ . Otherwise, set  $w(z_3v) = w(z_2v) = 1$ . With



$w(z_3v)$  and  $w(z_2v)$  fixed, apply  $S_2$  and  $S_3$ . Now choose  $w(v)$  and  $w(vz_1)$  to satisfy  $vz_2$  and  $vz_3$ , with  $w(v) \leq w(vz_1)$ .

If we have set  $w(z_3v) = w(z_2v) = 2$ , then  $w'(z_1x_1) = w'(z_1x'_1) = 1$ ; satisfy  $vz_1$  by setting  $w(z_1) = w(z_1y_1) = 1$ . Since  $\{a, a'\} = \{6, 7\}$ , this also satisfies  $\Gamma_{G'}(z_1)$ .

If we have set  $w(z_3v) = w(z_2v) = 1$ , then  $w'(z_1x_1) + w'(z_1x'_1) \geq 3$  or  $\{a, a'\} \neq \{6, 7\}$ . In the first case,  $vz_1$  is automatically satisfied; apply  $S_1$ . In the second,  $w(z_3v) = w(z_2v) = w'(z_1x_1) = w'(z_1x'_1) = 1$  and  $\{a, a'\} \neq \{6, 7\}$ ; choose  $b \in \{6, 7\} - \{a, a'\}$ . If  $w(v) = 1$ , then  $vz_1$  is automatically satisfied; apply  $S_1$ . Otherwise,  $w(v) = w(vz_1) = 2$ , since we chose  $w(v) \leq w(vz_1)$ . Now choose  $w(z_1)$  and  $w(z_1y_1)$  with sum  $b-3$ . This satisfies  $vz_1$  and  $\Gamma_{G'}(z_1)$ .

**Subcase 2:** Set  $w(z_3v) = 1$ . With  $w(z_3v)$  fixed, apply  $S_3$ . If  $w'(z_1x_1) = 2$ , then setting  $w(z_1v) = w(v) = 1$  ensures satisfying  $z_1v$  and  $z_2v$ ; choose  $w(z_2v)$  to satisfy  $vz_3$  and apply  $S_1$  and  $S_2$ . By symmetry, we may thus assume  $w'(z_ix_i) = w'(z_ix'_i) = 1$  for  $i \in \{1, 2\}$ .

If  $w(z_3) + w'(z_3x_3) + w(z_3y_3) > 3$ , then setting  $w(v) = w(vz_2) = w(vz_1) = 1$  satisfies  $vz_3$  and ensures satisfying  $vz_2$  and  $vz_1$ ; apply  $S_2$  and  $S_1$ . Hence we may also assume  $w(z_3) + w'(z_3x_3) + w(z_3y_3) = 3$ . Let  $a_i = \rho_{w'}(x_i, z_i)$  and  $a'_i = \rho_{w'}(x'_i, z_i)$ , for  $i \in \{1, 2\}$ .

If  $\{a_1, a'_1\} \neq \{5, 6\}$ , then set  $w(vz_1) = w(v) = 1$  and  $w(vz_2) = 2$ . Now  $vz_3$  is satisfied and  $vz_2$  is automatically satisfied; apply  $S_2$ . Choose  $b \in \{5, 6\} - \{a_1, a'_1\}$ . Choose  $w(z_1)$  and  $w(z_1y_1)$  summing to  $b-2$ . This satisfies  $vz_1$  and  $\Gamma_{G'}(z_1)$ .

By symmetry, we may thus assume  $\{a_1, a'_1\} = \{a_2, a'_2\} = \{5, 6\}$ . Let  $w(v) = 2$  and  $w(vz_i) = w(z_i) = w(z_iy_i) = 2$  for  $i \in \{1, 2\}$  to satisfy all remaining edges.

**Subcase 3:** Set  $w(v) = w(vz_1) = w(vz_2) = w(vz_3) = 1$  to guarantee satisfying each  $vz_i$ . Now for each  $i$  choose  $w(y_iz_i)$  and  $w(z_i)$  to satisfy  $z_ix_i$  and  $z_ix'_i$ .  $\square$

Case **F** and Case **G** in Lemma 3.4 can be generalized. If  $v$  in the former case or  $z_i$  in the latter is a  $\beta'$ -vertex of any even degree, then the configuration remains 2-reducible. We omit this since it is not needed to prove the 1, 2-Conjecture for  $\text{Mad}(G) < 8/3$ ; the more restrictive configurations in the lemma complete an unavoidable set.

**Lemma 3.5.** *If  $\bar{d}(G) < 8/3$ , then  $G$  contains one of the following:*

- A.** *A  $3^-$ -vertex having a 1-neighbor.*
- B.** *A  $4^-$ -vertex having two  $2^-$ -neighbors.*
- C.** *A  $5^+$ -vertex  $v$  whose number of  $2^-$ -neighbors is at least  $(d_G(v) - 1)/2$ .*
- D.** *Two adjacent  $\beta$ -vertices.*
- E.** *A  $\beta$ -vertex with a  $\beta'$ -neighbor.*
- F.** *A  $\beta'$ -vertex of degree 4 having two  $\beta'$ -neighbors of degree 4.*
- G.** *A 3-vertex such that each neighbor is a  $\beta$ -vertex or is a  $\beta'$ -vertex of degree 4.*

*Proof.* We prove that avoiding **A-G** requires  $\bar{d}(G) \geq 8/3$ . Give each  $v \in V(G)$  initial charge  $d_G(v)$ . Move charge via the following rules:

- (1) Each 1-vertex takes  $\frac{5}{3}$  from its neighbor.

- (2) Each 2-vertex takes  $\frac{2}{3}$  from one  $3^+$ -neighbor.
- (3) Each 3-vertex with a 2-neighbor takes  $\frac{1}{6}$  from each other neighbor.
- (4) Each 4-vertex with a 1-neighbor takes  $\frac{1}{6}$  from each other neighbor not a  $\beta'$ -vertex.

Let  $\mu(v)$  denote the resulting charge at  $v$ ; it suffices to check that  $\mu(v) \geq \frac{8}{3}$  for all  $v$ . For  $\mathbf{Z} \in \{\mathbf{A}, \dots, \mathbf{G}\}$ , let  $\overline{\mathbf{Z}}$  mean “configuration  $\mathbf{Z}$  does not occur in  $G$ ”. Note that by  $\overline{\mathbf{B}}$ , the vertex taking charge in Rule 3 or Rule 4 is a  $\beta$ -vertex or a  $\beta'$ -vertex, respectively.

Case  $d(v) = 1$ : By Rule 1,  $v$  receives  $\frac{5}{3}$  and has final charge  $\frac{8}{3}$ .

Case  $d(v) = 2$ : By  $\overline{\mathbf{A}}$  and  $\overline{\mathbf{B}}$ ,  $v$  gives no charge and receives  $\frac{2}{3}$  from a  $3^+$ -neighbor.

Case  $d(v) = 3$ : By  $\overline{\mathbf{A}}$ ,  $v$  has no 1-neighbor. If  $v$  also has no 2-neighbor, then by  $\overline{\mathbf{G}}$  at most two neighbors take charge  $\frac{1}{6}$  from it, so  $\mu(v) \geq \frac{8}{3}$ . If  $v$  has a 2-neighbor, then by  $\overline{\mathbf{B}}$  it is a  $\beta$ -vertex, has only one 2-neighbor, and may give  $\frac{2}{3}$  to that 2-neighbor. By  $\overline{\mathbf{D}}$  and  $\overline{\mathbf{E}}$ ,  $v$  loses no other charge. If  $v$  does loses  $\frac{2}{3}$ , then  $v$  needs to regain  $\frac{1}{3}$  and does so via Rule 3.

Case  $d(v) = 4$ : By  $\overline{\mathbf{B}}$ ,  $v$  has at most one  $2^-$ -neighbor. If  $v$  has no 1-neighbor, then  $v$  loses at most  $\frac{2}{3} + \frac{3}{6}$ . Hence in this case  $\mu(v) > \frac{8}{3}$ . If  $v$  has a 1-neighbor, then  $v$  loses  $\frac{5}{3}$  to it and is a  $\beta'$ -vertex. By  $\overline{\mathbf{E}}$  and  $\overline{\mathbf{F}}$ ,  $v$  has no  $\beta$ -neighbor and at most one  $\beta'$ -neighbor. Hence it gives away no other charge and receives at least  $\frac{2}{6}$  to reach  $\mu(v) \geq \frac{8}{3}$ .

Case  $d(v) \geq 5$ : By  $\overline{\mathbf{C}}$ ,  $v$  has at most  $\frac{d(v)-2}{2}$   $2^-$ -neighbors. If the inequality is strict, then  $v$  gives at most  $\frac{5}{3} \frac{d(v)-3}{2}$  to those vertices and at most  $\frac{1}{6}$  to each other neighbor, yielding

$$\mu(v) \geq d(v) - \frac{5}{3} \frac{d(v)-3}{2} - \frac{1}{6} \frac{d(v)+3}{2} = \frac{d(v)}{12} + \frac{9}{4} \geq \frac{32}{12} = \frac{8}{3}.$$

If  $v$  has exactly  $\frac{d(v)-2}{2}$   $2^-$ -neighbors and at least one of them is a 2-vertex, then  $d(v) \geq 6$  and

$$\mu(v) \geq d(v) - \frac{5}{3} \frac{d(v)-4}{2} - \frac{2}{3} - \frac{1}{6} \frac{d(v)+2}{2} = \frac{d(v)}{12} + \frac{8}{3} - \frac{1}{6} > \frac{8}{3}.$$

In the remaining case,  $v$  is a  $\beta'$ -vertex with degree at least 6. By definition,  $v$  has  $\frac{d(v)-2}{2}$  1-neighbors and no 2-neighbor. By  $\overline{\mathbf{E}}$ ,  $v$  has no  $\beta$ -neighbor. By Rules 3 and 4,  $v$  gives charge only to its 1-neighbors. Hence

$$\mu(v) \geq d(v) - \frac{5}{3} \frac{d(v)-2}{2} = \frac{d(v)}{6} + \frac{5}{3} \geq \frac{8}{3}. \quad \square$$

Since every configuration in Lemma 3.5 is listed in Lemma 3.4, the following is proved.

**Theorem 3.6.** *The 1,2-Conjecture holds for each graph  $G$  such that  $\text{Mad}(G) < 8/3$ .*

## 4 Proper 3-weighting when $\text{Mad}(G) < 8/3$

For the discussion of proper 3-weightings, again it will be helpful to have notation for special types of vertices. The definition of  $\beta$ -vertex is the same as before, but instead of  $\beta'$ -vertices we introduce  $\alpha$ -vertices and  $\gamma$ -vertices.

**Definition 4.1.** An  $\alpha$ -vertex is a 2-vertex with a 2-neighbor. A  $\beta$ -vertex is a 3-vertex with a 2-neighbor. A  $\gamma$ -vertex is a 4-vertex with a 1-neighbor or is a 3-vertex with an  $\alpha$ -neighbor or two 2-neighbors.

**Lemma 4.2.** *If  $\bar{d}(G) < 8/3$ , then  $G$  contains one of the following:*

- A.** A 2-vertex or 3-vertex having a 1-neighbor.
- B.** A  $4^-$ -vertex whose neighbors all have degree 2.
- C.** A 3-vertex having an  $\alpha$ -neighbor and another 2-neighbor.
- D.** A 4-vertex having a 1-neighbor and a  $2^-$ -neighbor.
- E.** A  $5^+$ -vertex  $v$  with  $3p_1 + 2p_2 \geq d(v)$ , where  $p_i$  is the number of  $i$ -neighbors of  $v$ .
- F.** Two adjacent  $\gamma$ -vertices.
- G.** A 3-vertex with two  $\gamma$ -neighbors.
- H.** A 6-vertex or 7-vertex having a 1-neighbor and four  $\gamma$ -neighbors.
- I.** A 5-vertex having a 1-neighbor and three  $\gamma$ -neighbors.
- J.** A 4-vertex with  $p + q + r \geq 5$ , where  $p, q, r$  are its numbers of 2-neighbors,  $\gamma$ -neighbors, and  $\alpha$ -neighbors, respectively.
- K.** A  $\gamma$ -vertex whose  $3^+$ -neighbors are all  $\beta$ -vertices.

*Proof.* We prove that avoiding **A-K** requires  $\bar{d}(G) \geq 8/3$ . Give each  $v \in V(G)$  initial charge  $d(v)$ . Move charge via the following rules:

- (1) Each 1-vertex takes  $\frac{5}{3}$  from its neighbor.
- (2) Each  $\alpha$ -vertex takes  $\frac{2}{3}$  from its  $3^+$ -neighbor.
- (3) Each 2-vertex that is not an  $\alpha$ -vertex takes  $\frac{1}{3}$  from each neighbor.
- (4) Each  $\gamma$ -vertex takes  $\frac{1}{3}$  from each  $3^+$ -neighbor that is not a  $\beta$ -vertex.

Let  $\mu(v)$  denote the resulting charge at  $v$ ; it suffices to check that  $\mu(v) \geq \frac{8}{3}$  for all  $v$ . For  $\mathbf{Z} \in \{\mathbf{A}, \dots, \mathbf{K}\}$ , let  $\bar{\mathbf{Z}}$  mean “configuration  $\mathbf{Z}$  does not occur in  $G$ ”. Fix a vertex  $v$  and let  $p_i, q$ , and  $r$  count its  $i$ -neighbors,  $\gamma$ -neighbors, and  $\alpha$ -neighbors, respectively.

Case  $d(v) = 1$ : By Rule 1,  $v$  receives  $\frac{5}{3}$  and has final charge  $\frac{8}{3}$ .

Case  $d(v) = 2$ : By  $\bar{\mathbf{A}}$ ,  $v$  gives no charge; via Rule 2 or Rule 3,  $v$  receives  $\frac{2}{3}$ .

Case  $d(v) = 3$ : By  $\bar{\mathbf{A}}$  and  $\bar{\mathbf{B}}$ ,  $v$  has no 1-neighbor and at most two 2-neighbors.

If  $v$  has no 2-neighbor, then  $v$  is not a  $\gamma$ -vertex or a  $\beta$ -vertex. It loses nothing via Rule 2 or 3, and via Rule 4 it gives  $\frac{1}{3}$  to each  $\gamma$ -neighbor. By  $\bar{\mathbf{G}}$ ,  $v$  loses at most  $\frac{1}{3}$ , and  $\mu(v) \geq \frac{8}{3}$ .

If  $v$  has a 2-neighbor, then  $v$  is a  $\beta$ -vertex and loses nothing via Rule 4. By  $\overline{\mathbf{B}}$  and  $\overline{\mathbf{C}}$ ,  $v$  gives at most  $\frac{2}{3}$  to its 2-neighbors, with equality only if it is a  $\gamma$ -vertex. In this case, by  $\overline{\mathbf{A}}$  and  $\overline{\mathbf{B}}$ ,  $v$  has a  $3^+$ -neighbor  $u$ . By  $\overline{\mathbf{F}}$ ,  $u$  is not a  $\gamma$ -vertex, and by  $\overline{\mathbf{K}}$  not all choices of  $u$  are  $\beta$ -vertices. Hence from some  $3^+$ -neighbor  $v$  receives  $\frac{1}{3}$ , and  $\mu(v) \geq \frac{8}{3}$ .

Case  $d(v) = 4$ : If  $v$  has a 1-neighbor, then  $v$  is a  $\gamma$ -vertex. By  $\overline{\mathbf{D}}$  it has no other  $2^-$ -neighbor and loses at most  $\frac{5}{3}$ . Its other neighbors are  $3^+$ -vertices. By  $\overline{\mathbf{F}}$ , none is a  $\gamma$ -vertex, and by  $\overline{\mathbf{K}}$  they are not all  $\beta$ -vertices. Hence  $v$  receives at least  $\frac{1}{3}$  via Rule 4, and  $\mu(v) \geq \frac{8}{3}$ .

If  $v$  has no 1-neighbor, then  $v$  is not a  $\gamma$ -vertex. By  $\overline{\mathbf{J}}$ ,  $p + q + r \leq 4$ . An  $\alpha$ -neighbor contributes to both  $p$  and  $r$ . Hence  $v$  loses exactly  $(p + q + r)/3$ , yielding  $\mu(v) \geq \frac{8}{3}$ .

Case  $d(v) \geq 5$ : The charge lost by  $v$  is  $\frac{1}{3}(5p_1 + p_2 + q + r)$ . Hence  $v$  is happy if  $5p_1 + p_2 + q + r \leq 3d(v) - 8$ . Also, configurations with  $3p_1 + 2p_2 \geq d(v)$  are forbidden, by  $\overline{\mathbf{E}}$ . Hence if  $v$  is not happy and is not in a configuration forbidden by  $\overline{\mathbf{E}}$ , then

$$5p_1 + p_2 + q + r \geq 3d(v) - 7 \quad \text{and} \quad 3p_1 + 2p_2 \leq d(v) - 1. \quad (1)$$

Since  $r \leq p_2$  and  $q \leq d(v) - p_1$ , the first inequality above yields  $4p_1 + 2p_2 \geq 2d(v) - 7$ . Eliminating  $2p_2$  from the two inequalities then yields  $2d(v) - 7 - 4p_1 \leq d(v) - 1 - 3p_1$ , which simplifies to  $d(v) - 6 \leq p_1$ . Since also  $p_1 \leq \lfloor (d(v) - 1)/3 \rfloor$ , we obtain  $d(v) \leq 8$ .

If  $p_1 = 0$ , then substituting  $q + r \leq d(v)$  in (1) yields  $(d(v) - 1)/2 \geq 2d(v) - 7$ , which simplifies to  $d(v) \leq 13/3$ . Hence we may assume  $p_1 \geq 1$ . We consider below the remaining unexcluded possibilities for  $(p_1, p_2, r, q)$ . In each case these are the choices allowed by (1).

For  $d(v) = 5$ , the remaining case is  $(1, 0, 0, q)$  with  $q \in \{3, 4\}$ , forbidden by  $\overline{\mathbf{I}}$ .

For  $d(v) = 6$ , the remaining case is  $(1, 1, 1, 4)$ , forbidden by  $\overline{\mathbf{H}}$ .

For  $d(v) = 7$ , the case  $(1, p_2, r, q)$  requires  $p_2 \leq 1$ , which yields  $5p_1 + 2p_2 + q \leq 12 < 14$ , so  $v$  remains happy. Hence the remaining case is  $(2, 0, 0, q)$  with  $q \in \{4, 5\}$ , forbidden by  $\overline{\mathbf{H}}$ .

For  $d(v) = 8$ , with  $p_1 \leq 2$  and  $3p_1 + 2p_2 \leq 7$ , we have  $5p_1 + 2p_2 \leq 10$ . At most  $16/3$  is lost, and hence  $\mu(v) \geq \frac{8}{3}$ .  $\square$

The next lemma explains the role of  $\gamma$ -vertices.

**Lemma 4.3.** *Let  $v$  be a  $\gamma$ -vertex having a  $3^+$ -neighbor  $x$ . Define  $F \subseteq E(G)$  as follows:*

*$F = \{vu\}$  if  $v$  is a 4-vertex with 1-neighbor  $u$ ,*

*$F = \{vz, vz'\}$  if  $v$  is a 3-vertex with 2-neighbors  $z$  and  $z'$ ,*

*$F = \Gamma_G(z)$  if  $v$  is a 3-vertex with  $\alpha$ -neighbor  $z$ .*

*Given any weighting of  $G - F$ , weights in  $\{1, 2, 3\}$  can be chosen on  $F$  to satisfy all edges in  $F$  or incident to edges of  $F$  except  $vx$ , without changing the weights on edges not in  $F$ .*

*Proof.* Figure 11 shows  $F$  in bold; the weight on  $vx$  is fixed. When  $v$  is a 4-vertex, choose  $w(vu)$  to satisfy the two edges from  $v$  to  $N_G(v) - \{x, u\}$ . When  $v$  is a 3-vertex with 2-neighbors  $z$  and  $z'$  having neighbors  $y$  and  $y'$  other than  $v$ , choose  $w(vz)$  to satisfy  $vz'$  and  $zy$ , and choose  $w(vz')$  to satisfy  $vz$  and  $z'y'$ . When  $v$  is a 3-vertex with  $\alpha$ -neighbor  $z$  having

neighbor  $y$  other than  $v$ , let  $x'$  and  $y'$  be the remaining neighbors of  $v$  and  $y$ . Choose  $w(vz)$  to satisfy  $vx'$  and  $zy$ , and choose  $w(zy)$  to satisfy  $vz$  and  $yy'$ .  $\square$

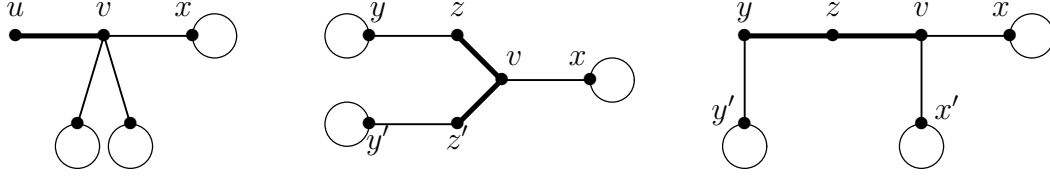


Figure 11: Three cases for Lemma 4.3, with  $F$  in bold

In employing Lemma 4.3, the difficulty is ensuring that the edge  $vx$  will be satisfied. Generally, we will need to ensure that some edge is satisfied regardless of the choice of weight on some incident edge. When  $v$  is a  $\gamma$ -vertex, let  $F_v$  denote the set of one or two edges designated as  $F$  in Lemma 4.3 (bold in Figure 11).

Like Lemma 2.5, the next lemma excludes degenerate cases for the reducibility proofs.

**Lemma 4.4.** *Let  $z$  and  $z'$  be  $\beta$ -vertices having respective 2-neighbors  $y$  and  $y'$  that are equal or adjacent. The following cases lead to 3-reducible configurations:*

- (1)  $zz' \in E(G)$ .
- (2)  $z$  and  $z'$  have a common neighbor  $v$  with a 1-neighbor  $u$ , such that  $d_G(v) \in \{4, 5\}$ .

*Proof.* See Figure 12 for these cases.

(1) Suppose  $zz' \in E(G)$ . If  $y = y'$ , then Lemma 2.5 applies. If  $yy' \in E(G)$ , then let  $G' = G - \{zz', zy, yy', z'y'\}$ . Set  $w(yy') = 1$  to ensure satisfying  $zy$  and  $z'y'$ . Set  $w(zy) = 1$ . Now choose  $w(z'y')$  to satisfy  $yy'$  and  $zz'$ , and choose  $w(zz')$  to satisfy  $zx$  and  $z'x'$ , where  $N_G(z) = \{z', y, x\}$  and  $N_G(z') = \{z, y', x'\}$  ( $x = x'$  is allowed).

(2) By Case (1), we may assume  $zz' \notin E(G)$ , so  $x \neq z'$  and  $x' \neq z$ .

(2a) If  $y = y'$ , then let  $G' = G - \{vz, vz', zy, z'y', vu\}$ . Set  $w(zy) = w(z'y') = 1$  to ensure satisfying  $zy$  and  $z'y'$ . Next choose  $w(zv) \in \{2, 3\}$  to satisfy  $zx$  and choose  $w(z'v) \in \{2, 3\}$  to satisfy  $z'x'$ . To ensure satisfying  $vz$  and  $vz'$ , we now choose  $w(vu)$  to satisfy  $\Gamma_{G'}(v)$  (if  $d_G(v) = 5$ ) or  $w(vu) \in \{2, 3\}$  to satisfy  $\Gamma_{G'}(v)$  (if  $d_G(v) = 4$ ).

(2b) If  $yy' \in E(G)$ , then let  $G' = G - \{vz, vz', zy, z'y', yy', vu\}$ . Set  $w(yy') = 1$  to ensure satisfying  $zy$  and  $z'y'$ . Set  $w(zy) = 1$  and  $w(vz') = 3$  to ensure satisfying  $zv$ . Next choose  $w(z'y')$  to satisfy  $yy'$  and  $z'x'$ . Two choices of  $w(vz)$  will satisfy  $zx$ . Along with the three choices available for  $w(vu)$  these choices can be made to satisfy  $vz'$  and  $\Gamma_{G'}(v)$ .  $\square$

**Lemma 4.5.** *The following configurations are 3-reducible.*

- A. A 2-vertex or 3-vertex having a 1-neighbor.
- B. A 4<sup>-</sup>-vertex whose neighbors all have degree 2.

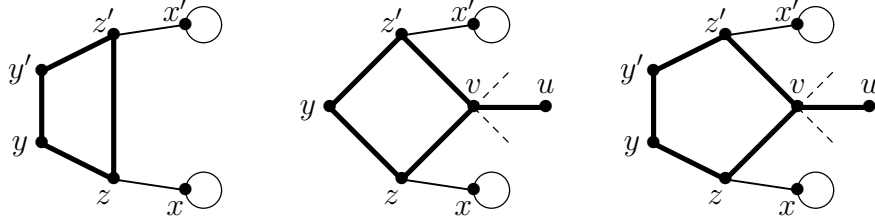


Figure 12: Three cases for Lemma 4.4

**C.** A 3-vertex having an  $\alpha$ -neighbor and another 2-neighbor.

**D.** A 4-vertex having a 1-neighbor and a  $2^-$ -neighbor.

**E.** A  $5^+$ -vertex  $v$  with  $3p_1 + 2p_2 \geq d(v)$ , where  $p_i$  is the number of  $i$ -neighbors of  $v$ .

**F.** Two adjacent  $\gamma$ -vertices.

**G.** A 3-vertex with two  $\gamma$ -neighbors.

**H.** A vertex  $v$  such that  $p_1 + 2q \geq d(v)$  and  $p_1 + q > 4$ , where  $p_1$  is the number of 1-neighbors and  $q$  is the number of  $\gamma$ -neighbors of  $v$ .

**I.** A 5-vertex having a 1-neighbor and three  $\gamma$ -neighbors.

**J.** A 4-vertex with (1) two  $\alpha$ -neighbors, (2) an  $\alpha$ -neighbor, another 2-neighbor, and a  $\gamma$ -neighbor, or (3) a 2-neighbor and three  $\gamma$ -neighbors.

**K.** A  $\gamma$ -vertex whose  $3^+$ -neighbors are all  $\beta$ -vertices.

*Proof.* Lemma 2.6 shows that **A-E** are 3-reducible. Let  $G$  be a minimal 3-bad graph containing one of **F-K**. When  $z$  is a  $\gamma$ -vertex with a  $3^+$ -neighbor  $v$  and  $w(zv)$  has been chosen, the phrase “apply Lemma 4.3 to  $z$ ” means “apply Lemma 4.3 to choose weights on  $F_z$  to satisfy  $F_z$  and the edges incident to them other than  $zv$ ”. In each case, we extend a proper 3-weighting  $w'$  of a proper subgraph  $G'$  of  $G$  to a proper 3-weighting  $w$  of  $G$ .

The phrase “Figure  $n$  is accurate” means that the vertices shown are known to be distinct, except possibly for non- $\gamma$ -vertices on circles, to which no edges of the core are incident. There are three types of  $\gamma$ -vertices. Let those of degree 4 be  $\gamma_4$ -vertices, those of degree 3 with an  $\alpha$ -neighbor be  $\gamma_{3a}$ -vertices, and those of degree 3 with two 2-neighbors be  $\gamma_{3b}$ -vertices.

**Case F:**  $v$  and  $v'$  are adjacent  $\gamma$ -vertices. Let  $G' = G - vv' - F_v - F_{v'}$ . By symmetry, the first subcase covers when  $v$  or  $v'$  is a  $\gamma_{3b}$ -vertex. By Lemma 2.5,  $v$  and  $v'$  do not have a common 2-neighbor.

**Subcase 1:**  $v$  is a  $\gamma_{3b}$ -vertex. Let  $N_G(v) = \{v', z, z'\}$ . By Lemma 2.5, we may assume  $zz', zv', z'v' \notin E(G)$ , so Figure 13 is accurate. Set  $w(vv') = 3$  to ensure satisfying  $vz$  and  $vz'$ . Apply Lemma 4.3 to  $v'$ . Choose  $w(vz)$  to satisfy  $\Gamma_{G'}(z)$ . Choose  $w(vz')$  to satisfy  $vv'$  and  $\Gamma_{G'}(z')$ .

**Subcase 2:**  $v$  and  $v'$  are both  $\gamma_{3a}$ -vertices. Let  $z$  and  $z'$  be the  $\alpha$ -neighbors of  $v$  and  $v'$ , and let  $y$  and  $y'$  be the 2-neighbors of  $z$  and  $z'$ . By Lemma 4.4, Lemma 2.5, and **B**, Figure 13 is accurate. Let  $x$  and  $x'$  be the remaining neighbors of  $v$  and  $v'$ , respectively.

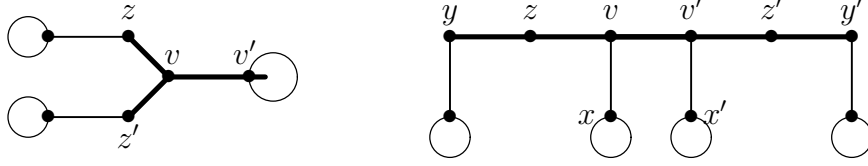


Figure 13: Cases **F1** and **F2** for Lemma 4.5

Choose  $w(vz)$  to satisfy  $zy$ . Now choose  $w(v'z')$  to satisfy  $vv'$  and  $z'y'$ . Next choose  $w(vv')$  to satisfy  $vx$  and  $v'x'$ . Finally, choose  $w(zy)$  to satisfy  $vz$  and  $\Gamma_{G'}(y)$ , and choose  $w(z'y')$  to satisfy  $v'z'$  and  $\Gamma_{G'}(y')$ .

**Subcase 3:**  $v$  and  $v'$  are both  $\gamma_4$ -vertices. Let  $N(v) = \{v', z_1, z_2, u\}$  and  $N(v') = \{v, z'_1, z'_2, u'\}$ , with  $d_G(u) = d_G(u') = 1$ . For  $i \in \{1, 2\}$ , let  $a_i = w'(vz_i)$  and  $b_i = \rho_{w'}(z_i, v)$ ; similarly define  $a'_i$  and  $b'_i$  using  $\{v', z'_1, z'_2\}$ . Since  $d_G(u), d_G(u') = 1$ , Figure 14 is accurate.

If  $a_1 + a_2 > a'_1 + a'_2$ , then set  $w(uv) = 3$  to ensure satisfying  $vv'$ . Next choose  $w(vv')$  to satisfy  $vz_1$  and  $vz_2$ , and choose  $w(v'u')$  to satisfy  $v'z'_1$  and  $v'z'_2$ .

By symmetry, we may thus assume  $a_1 + a_2 = a'_1 + a'_2$ . If  $b_1 \neq a_2 + 4$ , then choose  $w(vv') \in \{1, 3\}$  to ensure satisfying  $vz_1$ . Now choose  $w(v'u')$  to satisfy  $v'z'_1$  and  $v'z'_2$ , and then choose  $w(vu)$  to satisfy  $vz_2$  and  $vv'$ . Hence by symmetry we may assume that each entry of  $(b_1, b_2, b'_1, b'_2)$  exceeds the corresponding entry of  $(a_2, a_1, a'_2, a'_1)$  by exactly 4. Now setting  $w(uv) = w(vv') = 3$  and  $w(v'u') = 2$  completes the extension.

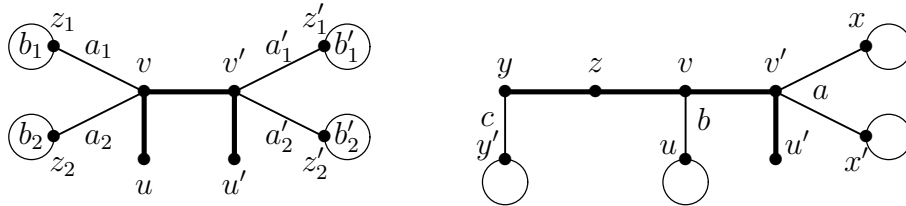


Figure 14: Cases **F3** and **F4** for Lemma 4.5

**Subcase 4:**  $v$  is a  $\gamma_{3a}$ -vertex and  $v'$  is a  $\gamma_4$ -vertex. Let  $N_G(v) = \{v', z, u\}$ , with  $z$  being the  $\alpha$ -vertex. Let  $y$  be the 2-neighbor of  $z$ , with  $N_G(y) = \{z, y'\}$ . Let  $N_G(v') = \{v, x, x', u'\}$ , with  $d_G(u') = 1$ . By Lemma 2.5 and **D**, Figure 14 is accurate. Let  $a = w'(v'x) + w'(v'x')$ ,  $b = w'(vu)$ , and  $c = w'(yy')$ .

If  $a \neq b$ , then choose  $w(u'v') \in \{1, 3\}$  to ensure satisfying  $vv'$ . Now choose  $w(vv')$  to satisfy  $v'x$  and  $v'x'$ , choose  $w(vz)$  to satisfy  $vu$  and  $yz$ , and choose  $w(zy)$  to satisfy  $vz$  and the other edge at  $y$ .

If  $a = b$ , then set  $w(u'v') = c$ . Now choose  $w(vv')$  to satisfy  $v'x$  and  $v'x'$ . Next choose  $w(vz)$  to satisfy  $vv'$ ,  $yz$ , and  $uv$ , which succeeds because  $vv'$  and  $yz$  both are satisfied if and only if  $w(vz) \neq c$ . Finally choose  $w(zy)$  to satisfy  $vz$  and the other edge at  $y$ .

**Case G:**  $v$  is a 3-vertex having two  $\gamma$ -neighbors  $z$  and  $z'$ . Let  $N_G(v) = \{z, z', x\}$ . In each case, we will let  $G' = G - \{vz, vz'\} - F_z - F_{z'}$  (and  $w'$  is a proper 3-weighting of  $G'$ ). By **F**,  $zz' \notin E(G)$ . By Lemma 2.5, neither  $z$  nor  $z'$  shares a 2-neighbor with  $v$  or has a 2-neighbor adjacent to a 2-neighbor of  $v$ . Hence Figures 15 and 16 are accurate.

**Subcase 1:**  $z$  is a  $\gamma_{3a}$ -vertex. Let  $N_G(z) = \{v, y, u\}$ , with  $y$  the  $\alpha$ -neighbor of  $z$ . Let  $y'$  be the 2-neighbor of  $y$ , with  $N_G(y') = \{y, y''\}$ . Let  $a = w'(vx)$ ,  $b = w'(zu)$ , and  $c = w'(y'y'')$ , as shown on the left in Figure 15.

If  $b \neq a$ , then choose  $w(vz') \in \{1, 3\}$  to ensure satisfying  $vz$ . With  $w(vz')$  known, apply Lemma 4.3 to  $z'$ . Next choose  $w(vz)$  to satisfy  $vx$  and  $vz'$ . With  $vz$  automatically satisfied and  $w(vz)$  chosen, it now suffices to apply Lemma 4.3 to  $z$ .

Hence we may assume  $b = a$ . In this case, set  $w(vz') = c$  and apply Lemma 4.3 to  $z'$ . Choose  $w(vz)$  to satisfy  $vx$  and  $vz'$ . Now choose  $w(zy)$  to satisfy  $vz$ ,  $zu$ , and  $yy'$ ; this succeeds because  $yy'$  and  $vz$  both forbid  $w(zy) = c$ , so at most two choices for  $w(zy)$  are forbidden. Finally, choose  $w(yy')$  to satisfy  $zy$  and  $y'y''$ .

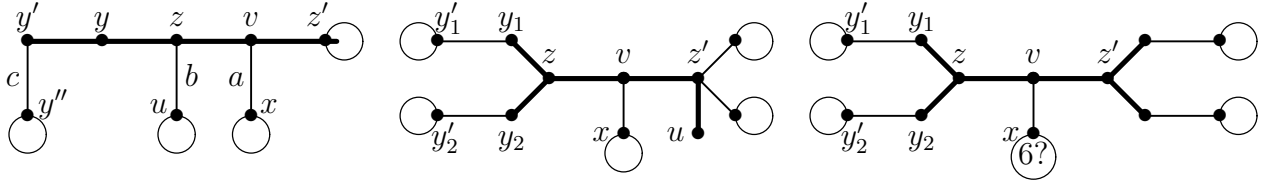


Figure 15: Cases **G1** and **G2** for Lemma 4.5

**Subcase 2:**  $z$  is a  $\gamma_{3b}$ -vertex. Let  $N_G(z) = \{v, y_1, y_2\}$ , with  $N_G(y_1) = \{z, y'_1\}$  and  $N_G(y_2) = \{z, y'_2\}$ . By Subcase 1, we may assume that  $z'$  is not a  $\gamma_{3a}$ -vertex.

Suppose that  $z'$  is a  $\gamma_4$ -vertex, with 1-neighbor  $u$ , as in Figure 15 in the middle. Set  $w(vz) = 3$  to ensure satisfying  $zy_1$  and  $zy_2$ . Now  $w(uz')$  has two choices that satisfy  $vz'$ , and  $w(vz')$  has two choices that satisfy  $vx$ . With at least three choices for the sum  $w(uz') + w(vz')$ , the two edges in  $\Gamma_{G'}(z')$  can also be satisfied. Now choose  $w(zy_1)$  to satisfy  $\Gamma_{G'}(y_1)$  and  $w(zy_2)$  to satisfy  $zv$  and  $\Gamma_{G'}(y_2)$ .

Hence we may assume that  $z'$  is also a  $\gamma_{3b}$ -vertex, as on the right in Figure 15. If  $\rho_{w'}(x, v) \neq 6$ , then set  $w(vz) = w(vz') = 3$ . This satisfies  $vx$  and also ensures satisfying  $F_z$  and  $F_{z'}$ . Choose  $w(zy_1)$  to satisfy  $y_1y'_1$ , and choose  $w(zy_2)$  to satisfy  $y_2y'_2$  and  $zv$ . Choose weights on  $F_{z'}$  by the same method.

If  $\rho_{w'}(x, v) = 6$  and  $w'(y_1y'_1) \neq 3$ , then set  $w(vz) = 2$  and  $w(vz') = 3$  to satisfy  $vx$  and ensure satisfying  $zy_1$ . Choose  $w(zy_1) \in \{2, 3\}$  to satisfy  $y_1y'_1$  and ensure satisfying  $zy_2$ . Now choose  $w(zy_2)$  to satisfy  $y_2y'_2$  and  $vz$ . Since setting  $w(vz') = 3$  ensures satisfying  $F_{z'}$ , we can choose weights on  $F_{z'}$  to finish as in the preceding paragraph.

Hence we may assume that  $\rho_{w'}(x, v) = 6$  and (by symmetry) that all the edges of  $G'$  incident to the 2-neighbors of  $z$  and  $z'$  have weight 3 under  $w'$ . Since we may assume by



Subcase 1 that neither  $z$  nor  $z'$  is a  $\gamma_{3a}$ -vertex, we can complete the extension by giving all the missing edges weight 1, unless  $w'(vx) = 1$ . In that case, just change  $w(vz)$  to 3 and choose  $w(zy_i) \in \{2, 3\}$  to satisfy  $y_i y'_i$ , for  $i \in \{1, 2\}$ .

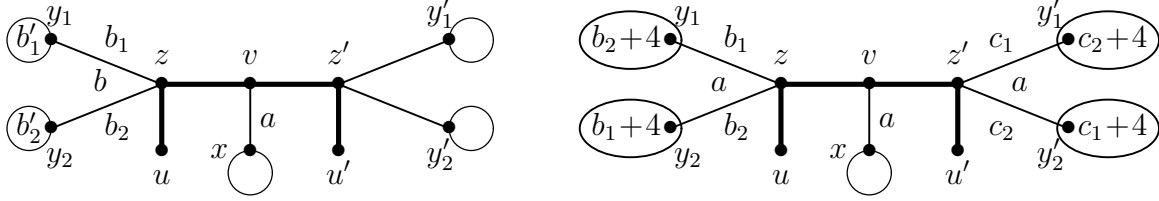


Figure 16: Case **G3** for Lemma 4.5

**Subcase 3:**  $z$  and  $z'$  are both  $\gamma_4$ -vertices. Let  $N_G(z) = \{v, y_1, y_2, u\}$  and  $N_G(z') = \{v, y'_1, y'_2, u'\}$ , with  $d_G(u) = d_G(u') = 1$ . Let  $a = w'(vx)$ . For  $i \in \{1, 2\}$ , let  $b_i = w'(zy_i)$  and  $b'_i = \rho_{w'}(y_i, z)$ , as on the left in Figure 16.

Let  $b = b_1 + b_2$ . If  $b \neq a$ , then choose  $w(vz') \in \{1, 3\}$  to ensure satisfying  $vz$ . With  $w(vz')$  fixed, choose  $w(z'u')$  to satisfy  $\Gamma_{G'}(z')$ . Now choose  $w(vz)$  to satisfy  $vx$  and  $vz'$ , and then choose  $w(zu)$  to satisfy  $\Gamma_{G'}(z)$  and complete the extension.

If  $b'_1 \neq b_2 + 4$ , then choose  $w(vz) \in \{1, 3\}$  to ensure satisfying  $zy_1$ . Now restrict  $w(u'z')$  to two choices that satisfy  $vz'$ , and restrict  $w(vz')$  to two choices that satisfy  $vx$ . Having at least three choices for the sum  $w(u'z') + w(vz')$  allows also satisfying the edges of  $\Gamma_{G'}(z')$ . Finally, choose  $w(uz)$  to satisfy  $zy_2$  and  $zv$  and complete the extension.

By symmetry, the only remaining case is  $b = a$ ,  $b'_1 = b_2 + 4$ , and  $b'_2 = b_1 + 4$ , and similarly for  $\Gamma_{G'}(z')$ , as on the right in Figure 16. Now  $w(vz) + w(zu) < 4$  satisfies  $zy_1$  and  $zy_2$ , and  $w(uz) \neq w(vz')$  satisfies  $vz$ . Similarly,  $w(vz') + w(z'u') > 4$  satisfies  $z'y'_1$  and  $z'y'_2$ , and  $w(u'z') \neq w(vz)$  satisfies  $vz'$ . Set  $w(vz) = w(zu) = 1$  and  $w(z'u') = 3$ , and choose  $w(vz') \in \{2, 3\}$  to satisfy  $vx$ .

**Case H:** A vertex  $v$  such that  $p_1 + 2q \geq d(v)$  and  $p_1 + q > 4$ . Let  $Z$  be the set of  $\gamma$ -neighbors of  $v$ . Let  $R$  be the set of edges from  $v$  to 1-neighbors and to  $Z$ , shown bold in Figure 17. By **F**, the set  $Z$  is independent. Form  $G'$  from  $G$  by deleting  $R$  and  $F_z$  for each  $z \in Z$ . Here  $v$  and a  $\gamma$ -neighbor of  $v$  play the roles of  $x$  and  $v$  in Figure 11, respectively.

Let  $R'$  be a set of  $d(v) - p_1 - q$  edges from  $v$  to  $Z$ . Assign weight 3 to all of  $R - R'$ . Choose weights on  $R'$  from  $\{2, 3\}$  to satisfy the  $d(v) - p_1 - q$  edges in  $\Gamma_{G'}(v)$ .

Consider  $vz$  with  $z \in Z$ . Including the weights on  $\Gamma_{G'}(v)$ , the sum of the weights on  $\Gamma_G(v) - \{vz\}$  is now at least  $3(q - 1) + 3p_1$ , which by hypothesis exceeds 9. At most three edges are incident to  $vz$  at  $z$ , so  $vz$  is automatically satisfied, as are the edges from  $v$  to 1-neighbors. Now the weights on  $R$  are fixed; apply Lemma 4.3 to the vertices of  $Z$ .

**Case I:** A 5-vertex  $v$  having a 1-neighbor and three  $\gamma$ -neighbors. The argument of Case **H** does not suffice here, since  $p_1 + q = 4$ . Let  $u$  be the 1-neighbor of  $v$ , and let  $z_1, z_2, z_3$  be

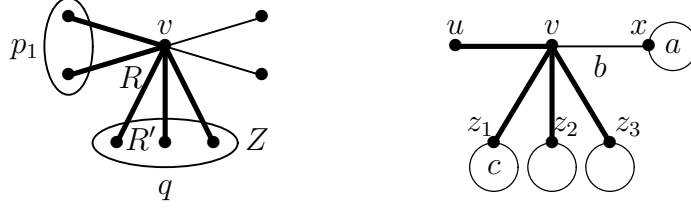


Figure 17: Cases **H** and **I** for Lemma 4.5

its  $\gamma$ -neighbors. As in **H**, let  $R = \{vu, vz_1, vz_2, vz_3\}$  (bold on the right in Figure 17), and let  $G' = G - R - \bigcup_i F_{z_i}$ . Let  $a = \rho_{w'}(x, v)$ , and let  $b = w'(vx)$ . By Lemma 4.4, we may assume that no two of the  $\gamma$ -neighbors are 3-vertices with a common 2-neighbor, and by **F** they form an independent set. Hence Figure 17 is accurate.

If  $a \neq 12$ , then put weight 3 on all edges of  $R$  to satisfy  $vx$  and ensure satisfying  $\{vz_1, vz_2, vz_3\}$ . Finally, apply Lemma 4.3 to each  $z_i$ .

If  $a = 12$ , then set  $w(vz_1) = 2$  so that  $vx$  is automatically satisfied. Having specified  $w(vz_1)$ , apply Lemma 4.3 to  $z_1$ . Now let  $c = \rho_w(z_1, v)$ . If  $c \leq 7$ , or if  $c = 8$  and  $b \geq 2$ , then set  $w(vz_2) = w(vz_3) = 3$  to ensure satisfying  $vz_1$ . If  $c = 9$ , or if  $c = 8$  and  $b = 1$ , then set  $w(vz_2) = w(vz_3) = 1$  to ensure satisfying  $vz_1$ . Next apply Lemma 4.3 to  $z_2$  and  $z_3$ . Finally, choose  $w(vu)$  to satisfy  $vz_2$  and  $vz_3$ .

**Case J:** A 4-vertex with specified neighbors. As usual, by the prior lemmas and reducible configurations, Figure 18 is accurate.

**Subcase 1:** A 4-vertex  $v$  with  $\alpha$ -neighbors  $z$  and  $z'$ . Let  $N_G(z) = \{v, y\}$  and  $N_G(z') = \{v, y'\}$ , as in Figure 18. By Lemma 2.5, we may assume  $\{y, y'\} \cap \{z, z'\} = \emptyset$ . By **B**, we have  $y \neq y'$  and  $yy' \notin E(G)$ . Let  $G' = G - \{vz, vz', zy, z'y'\}$ . At least two choices for  $w(vz)$  satisfy  $zy$ , and similarly two choices for  $w(vz')$  satisfy  $z'y'$ . This yields at least three choices for  $w(vz) + w(vz')$ , which is enough to satisfy  $\Gamma_{G'}(v)$ . Finally, choose  $w(zy)$  to satisfy its two incident edges and  $w(z'y')$  to satisfy its two incident edges.

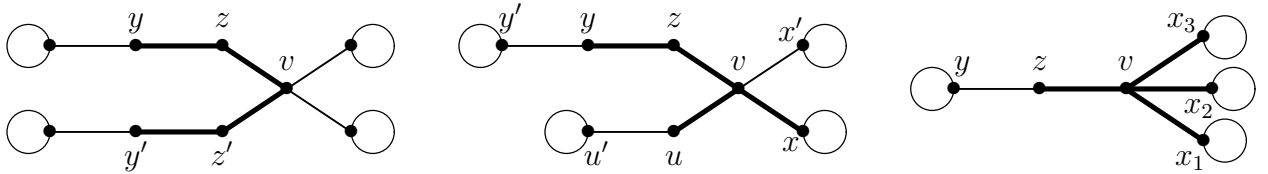


Figure 18: Cases **J1, J2, J3** for Lemma 4.5

**Subcase 2:** A 4-vertex  $v$  with an  $\alpha$ -neighbor  $z$ , another 2-neighbor  $u$  (that is not an  $\alpha$ -vertex), and a  $\gamma$ -neighbor  $x$ . Name the vertices (uniquely) so that  $y', y, z, v, u, u'$  form a path in order, and let  $x'$  be the remaining neighbor of  $v$ , as in Figure 18. Let  $G' = G - \{yz, zv, vu, vx\} - F_x$ . Set  $w(vx) = 2$  to ensure satisfying  $vu$ . Apply Lemma 4.3 to  $x$ . At

least two choices for  $w(vu)$  satisfy  $uu'$ , and at least two choices for  $w(vz)$  satisfy  $yz$ . With at least three choices for  $w(vu) + w(vz)$ , at least one satisfies  $vx$  and  $vx'$ . Finally, choose  $w(yz)$  to satisfy  $vz$  and  $yy'$ .

**Subcase 3:** *A 4-vertex  $v$  with a 2-neighbor  $z$  and three  $\gamma$ -neighbors  $x_1, x_2, x_3$ .* Let  $N(z) = \{v, y\}$ , as in Figure 18. Let  $G' = G - \Gamma_G(v) - \bigcup_i F_{x_i}$ .

If  $d(x_1) = 3$ , or if  $d(x_1) = 4$  and  $\phi_{w'}(x_1) \leq 4$ , then the value of  $\rho_w(x_1, v)$  will be at most 7, since when  $d(x_1) = 4$  there is only one edge in  $F_{x_1}$  (the edge incident to the 1-neighbor of  $x_1$ ). Hence setting  $w(vx_2) = w(vx_3) = 3$  and restricting  $w(zv)$  to  $\{2, 3\}$  ensures satisfying  $vx_1$  and  $vz$ . Apply Lemma 4.3 to  $x_2$  and  $x_3$ . Now choose  $w(zv)$  (in  $\{2, 3\}$ ) to satisfy  $yz$ , and then choose  $w(vx_1)$  to satisfy  $vx_2$  and  $vx_3$ . Finally, apply Lemma 4.3 to  $x_1$ .

By symmetry, we may now assume  $d(x_i) = 4$  and  $\phi_{w'}(x_i) \geq 5$  for all  $i$ . Choose  $w(vz) \in \{1, 2\}$  to satisfy  $zy$ . Set  $w(vx_1) = 2$  and  $w(vx_2) = w(vx_3) = 1$  to ensure satisfying all edges incident to  $v$ . Finally, apply Lemma 4.3 to each  $x_i$ .

**Case K:**  *$v$  is a  $\gamma$ -vertex whose  $3^+$ -neighbors are all  $\beta$ -vertices.* See Figure 19. Let  $S$  be the set of neighbors of  $v$  whose degrees are not specified by the definition of  $v$  being a  $\gamma$ -vertex. By **{B,C,D}**, all vertices of  $S$  are  $3^+$ -vertices. By **F**, they cannot be  $\gamma$ -vertices, so by the hypothesis of this case, each vertex of  $S$  has degree 3, with one 2-neighbor and one  $3^+$ -neighbor other than  $v$  (by **A**). By Lemmas 2.5 and 4.4 and Cases **B** and **F**, Figure 19 is accurate in each subcase.

**Subcase 1:**  *$v$  is a  $\gamma_{3b}$ -vertex.* Let  $N_G(v) = \{z, z', u\}$ , where  $d_G(u) = 3$ . Let  $y$  be the 2-neighbor of  $u$ . Let  $G' = G - \Gamma_G(v) - uy$ . Set  $w(vu) = 3$  to ensure satisfying all of  $\{vz, vz', uy\}$ . Choose  $w(uy)$  to satisfy its two incident edges other than  $vu$ . Choose  $w(vz)$  to satisfy the other edge at  $z$ , and choose  $w(vz')$  to satisfy  $vu$  and the other edge at  $z'$ .

**Subcase 2:**  *$v$  is a  $\gamma_4$ -vertex.* Let  $N_G(v) = \{z_1, z_2, z_3, u\}$ , with  $d_G(u) = 1$ . Let  $y_i$  be the 2-neighbor of  $z_i$ . Let  $G' = G - \Gamma_G(v) - \{z_1y_1, z_2y_2, z_3y_3\}$ . For each  $i$ , set  $w(vz_i) = 3$  to ensure satisfying  $z_iy_i$ , and choose  $w(z_iy_i)$  to satisfy its two incident edges other than  $z_iy_i$ . Also  $vz_1, vz_2, vz_3$  are satisfied.

**Subcase 3:**  *$v$  is a  $\gamma_{3a}$ -vertex.* Let  $u$  be the  $\alpha$ -neighbor of  $v$  (with  $N_G(u) = \{v, u'\}$ ). By **C**,  $v$  does not have another 2-neighbor. Let  $N_G(z) = \{v, x, y\}$  and  $N_G(z') = \{v, x', y'\}$ , with  $d_G(y) = d_G(y') = 2$ . By **J**, each  $\beta$ -neighbor of  $v$  is not a  $\gamma$ -vertex, which means that the other neighbors of  $y$  and  $y'$  are  $3^+$ -vertices.

Let  $G' = G - \Gamma_G(v) - \{uu', zy, z'y'\}$ . Let  $a$  and  $b$  be the weights under  $w'$  of the edges incident to  $y$  and  $z$  in  $G'$ , respectively. Set  $w(vz') = 3$  to ensure satisfying  $z'y'$ , and choose  $w(z'y')$  to satisfy the edges incident to  $z'y'$  other than  $vz'$ .

If  $a \leq b$ , then  $zy$  is automatically satisfied. Choose  $w(zy)$  to satisfy  $\Gamma_{G'}(y)$ . Now choose  $w(vu)$  to satisfy  $vz$  and  $uu'$ , and then choose  $w(vz)$  to satisfy  $vz'$  and  $zx$ . Finally, choose  $uu'$  to satisfy  $vu$  and  $u'u''$ .

If  $a > b$ , then setting  $w(zy) = 1$  ensures satisfying the other edge at  $y$  (since its other

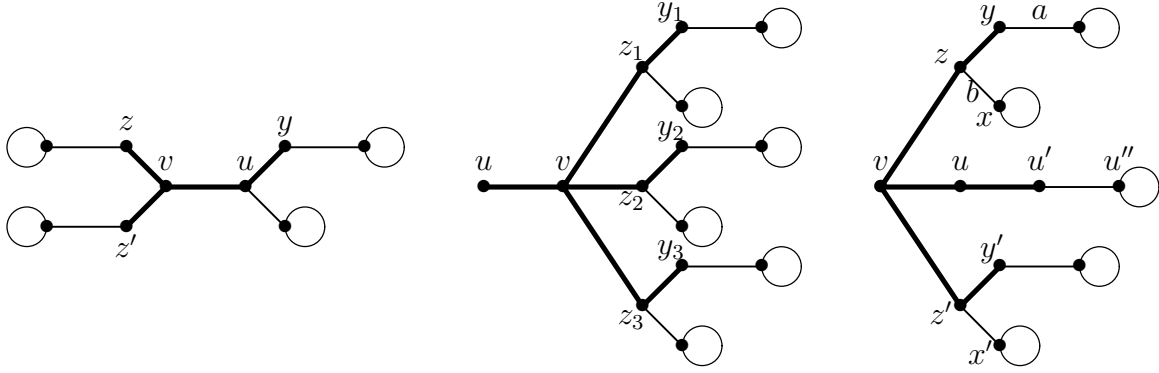


Figure 19: Cases **K1**, **K2**, **K3** for Lemma 4.5

endpoint has degree at least 3). With  $b \leq 2$  and  $w(vz') = 3$ , the edge  $vz$  is automatically satisfied. Now choose  $w(vz)$  to satisfy the other edges at  $z$ , choose  $w(vu)$  to satisfy  $vz'$  and  $uu'$ , and choose  $w(uu')$  to satisfy its incident edges.  $\square$

**Theorem 4.6.** *Every graph  $G$  with  $\text{Mad}(G) < \frac{8}{3}$  has a proper 3-weighting.*

*Proof.* It suffices to show that every configuration in the unavoidable set in Lemma 4.2 is shown to be 3-reducible in Lemma 4.5. The configurations are the same in the two lemmas except for **H** and **J**.

For **H**, if  $d(v) \in \{6, 7\}$  and  $v$  has a 1-neighbor and four  $\gamma$ -neighbors, then  $p_1 + 2q \geq d(v)$  and  $p_1 + q > 4$ . For **J**, a 4-vertex  $v$  with  $p + q + r \geq 5$  must have an  $\alpha$ -neighbor. If  $v$  has another  $\alpha$ -neighbor, then **J1** applies. If  $v$  has a  $\gamma$ -neighbor and another 2-neighbor, then **J2** applies. Otherwise, all other neighbors are  $\gamma$ -neighbors (reducible by **J3**) or all are 2-neighbors (reducible by **B**).  $\square$

Some of the 3-reducible configurations in Lemma 4.5 are more general than the configurations forced in Lemma 4.2. Also, there are other 3-reducible configuration we have not used, such as (1) a 4-vertex having two 2-neighbors and one  $\gamma$ -neighbor and (2) a more general version of configuration **H**. This suggests that with more work this approach could be pushed to prove the conclusion under a weaker restriction on  $\text{Mad}(G)$ .

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