1. **Theorem.** (Matrix Tree Theorem) Given a loopless graph $G$ with vertex set $v_1, \ldots, v_n$, let $a_{i,j}$ be the number of edges with endpoints $v_i$ and $v_j$. Let $Q(G)$ be the matrix $D - A$, where $A$ is the adjacency matrix of $G$ and $D$ is the diagonal matrix with $d(v_i)$ in position $(i, i)$ for each $i$. If $Q^*(G)$ is the matrix obtained by deleting row $s$ and column $t$ of $Q(G)$, then $\tau(G) = \det Q^*(G)$.

**Proof:** (Bollobás [1998, p. 57]) We use induction on $e(G)$. If $e(G) = 0$ and $n(G) > 1$, then $G$ has no spanning tree and $Q^*(G)$ is an all-zero matrix of order at least 1, which has determinant 0. If $e(G) = 0$ and $n(G) = 1$, then $G$ has one spanning tree and $Q^*(G)$ is a 0-by-0 matrix, which by convention has determinant 1.

Now consider $e(G) > 0$. By renumbering vertices, we may assume that $s = 1$ and that $v_1 \leftrightarrow v_2$. Let $e$ be an edge with endpoints $v_1$ and $v_2$, and let $d_i = d_G(v_i)$. The matrices $Q(G - e)$ and $Q(G \cdot e)$ are similar to $Q(G)$. In fact, they all have the same submatrix in the last $n(G) - 2$ rows and columns; call this submatrix $R$. Let $P$ denote the part of the second row of $Q(G)$ after the second column; this part of $Q(G - e)$ is the same. Let $d'$ denote the degree in $G \cdot e$ of the contracted vertex, and let $P'$ be the remainder of its row in $Q(G \cdot e)$.

By the induction hypothesis, $\tau(G - e) = Q^*(G - e)$ and $\tau(G \cdot e) = Q^*(G \cdot e)$. By Proposition 2.2.8, these sum to $\tau(G)$. Let $d' = d_1 - 1$, and let $a = -a_{12}$ and $a' = -(a_{12} - 1)$. The steps in the computation appear below. In each matrix, the row and column deleted before taking the determinant are shaded.

\[
\tau(G) = \tau(G - e) + \tau(G \cdot e) = \det Q^*(G - e) + \det Q^*(G \cdot e)
\]

\[
= \det Q^*(G).
\]

The equality in the determinant computation uses the expansion formula along the row containing $P$. For the terms involving $P$, the two large determinants contribute the same, since the entry in $P$ is the same and the entries below this row are the same. For the first position, the computation from $G - e$ yields $(d_2 - 1) \det R$. Added to this is the contribution $\det R$ from $G \cdot e$. Hence the sum of all the contributions is precisely equal to $\det Q^*(G)$, as desired.

The full statement of the Matrix Tree Theorem allows deletion of any row and column: $\tau(G) = (-1)^{s+t} \det Q^*$ when the submatrix obtained by deleting row $s$ and column $t$ from $Q$ is $Q^*$. This follows from a lemma in linear algebra stating that when every row and column has sum 0, the cofactors are all equal; see Exercise 8.6.18.