1. Theorem. (Matrix Tree Theorem) Given a loopless graph G with vertex set $v_1, \dots, v_n$, let $a_{i,j}$ be the number of edges with endpoints v_i and v_j. Let $Q(G)$ be the matrix $D - A$, where A is the adjacency matrix of G and D is the diagonal matrix with $d(v_i)$ in position (i,i) for each i. If $Q^*(G)$ is the matrix obtained by deleting row s and column s of $Q(G)$, then $\tau(G) = \det Q^*(G)$.

Proof: (Bollobás [1998, p. 57]) We use induction on $e(G)$. If $e(G) = 0$ and $n(G) > 1$, then G has no spanning tree and $Q^*(G)$ is an all-zero matrix of order at least 1, which has determinant 0. If $e(G) = 0$ and $n(G) = 1$, then G has one spanning tree and $Q^*(G)$ is a 0-by-0 matrix, which by convention has determinant 1.

Now consider $e(G) > 0$. By renumbering vertices, we may assume that $s = 1$ and that $v_1 \leftrightarrow v_2$. Let e be an edge with endpoints v_1 and v_2, and let $d_i = d_G(v_i)$. The matrices $Q(G-e)$ and $Q(G \cdot e)$ are similar to $Q(G)$. In fact, they all have the same submatrix in the last $n(G) - 2$ rows and columns; call this submatrix R. Let P denote the part of the second row of $Q(G)$ after the second column; this part of $Q(G-e)$ is the same. Let d' denote the degree in $G \cdot e$ of the contracted vertex, and let P' be the remainder of its row in $Q(G \cdot e)$.

By the induction hypothesis, $\tau(G-e) = Q^*(G-e)$ and $\tau(G \cdot e) = Q^*(G \cdot e)$. By Proposition 2.2.8, these sum to $\tau(G)$. Let $d_i^- = d_i - 1$, and let $a = -a_{1,2}$ and $a^- = -(a_{1,2} - 1)$. The steps in the computation appear below. In each matrix, the row and column deleted before taking the determinant are shaded.

$$\tau(G) = \tau(G-e) + \tau(G \cdot e) = \det Q^*(G-e) + \det Q^*(G \cdot e)$$

$$= \begin{vmatrix} d_1^- & a^- & \\ a^- & d_2^- & P \\ & P^T & R \end{vmatrix} + \begin{vmatrix} d' & P' \\ P'^T & R \end{vmatrix} = \begin{vmatrix} d_1 & a & \\ a & d_2 & P \\ & P^T & R \end{vmatrix}$$

$$= \det Q^*(G).$$

The equality in the determinant computation uses the expansion formula along the row containing P. For the terms involving P, the two large determinants contribute the same, since the entry in P is the same and the entries below this row are the same. For the first position, the computation from $G - e$ yields $(d_2 - 1)\det R$. Added to this is the contribution $\det R$ from $G \cdot e$. Hence the sum of all the contributions is precisely equal to $\det Q^*(G)$, as desired. ∎

The full statement of the Matrix Tree Theorem allows deletion of any row and column: $\tau(G) = (-1)^{s+t} \det Q^*$ when the submatrix obtained by deleting row s and column t from Q is Q^*. This follows from a lemma in linear algebra stating that when every row and column has sum 0, the cofactors are all equal; see Exercise 8.6.18.