MATH 247 HONORS, FALL 1999 - PROBLEM SET 6

Do five of the following six problems. Due Wednesday, Oct. 6.
Test #1 Wednesday, Sept. 29, 7:30-9:30PM, 241 Altgeld Hall.

1. *Integer equations.*
a) Count the solutions in nonnegative integers $x_1, \dots, x_k$ to $x_1 + \cdots + x_k \le n$.
b) Count the solutions in positive integers $x_1, \dots, x_k$ to $x_1 + \cdots + x_k = n$.

2. By counting a set in two ways, prove that $\sum_{i=1}^{n}(i-1)(n-i) = \binom{n}{3}$.

3. Let A_n be the set of permutations of $[n]$. Let B_n be the set of n-tuples $(b_1, \dots, b_n)$ such that $1 \le b_i \le i$ for each $i \in [n]$. Construct a bijection from A_n to B_n. (Hint: use induction on n, employing a bijection from A_{n-1} to B_{n-1} to construct a bijection from A_n to B_n. Below we illustrate this process for $n = 3$.)

A_3	321	231	213	312	132	123
B_3	111	112	113	121	122	123

4. Let $\langle a \rangle$ be a sequence such that $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n + 2a_{n-1}$ for $n \ge 2$. Prove that a_n is divisible by 3 if and only if n is divisible by 3.

5. *Primes and non-primes.*
a) Prove using contradiction that the set of prime numbers is not finite.
b) For each $n \in \mathbb{N}$, construct a set of n consecutive positive integers that are not prime. (Hint: determine a positive integer x such that x is divisible by 2, $x+1$ is divisible by 3, $x+2$ is divisible by 4, etc.)

6. Let p be a prime number.
a) Prove that p divides $\binom{p}{k}$ if $1 \le k \le p-1$.
b) Prove that $n^p - n$ is divisible by p for every $n \in \mathbb{N}$. (Hint: use the binomial theorem and part (a) in a proof by induction.)

PROBLEMS FOR CLASS DISCUSSION

Pair 1 - 5.48, 6.9, 6.15.
Pair 2 - 5.49, 6.7, 6.16.
Pair 3 - 5.50, 6.3, 6.18.