MATH 247 HONORS, FALL 1999 - PROBLEM SET 6

Do five of the following six problems. Due Wednesday, Oct. 6.
Test #1 Wednesday, Sept. 29, 7:30-9:30PM, 241 Altgeld Hall.

1. Integer equations.
   a) Count the solutions in nonnegative integers $x_1, \ldots, x_k$ to $x_1 + \cdots + x_k \leq n$.
   b) Count the solutions in positive integers $x_1, \ldots, x_k$ to $x_1 + \cdots + x_k = n$.

2. By counting a set in two ways, prove that $\sum_{i=1}^{n} (i-1)(n-i) = \binom{n}{3}$.

3. Let $A_n$ be the set of permutations of $[n]$. Let $B_n$ be the set of $n$-tuples $(b_1, \ldots, b_n)$ such that $1 \leq b_i \leq i$ for each $i \in [n]$. Construct a bijection from $A_n$ to $B_n$. (Hint: use induction on $n$, employing a bijection from $A_{n-1}$ to $B_{n-1}$ to construct a bijection from $A_n$ to $B_n$. Below we illustrate this process for $n = 3$.)

\[
\begin{array}{c|ccc|ccc}
A_3 & 321 & 231 & 213 & 312 & 132 & 123 \\
B_3 & 111 & 112 & 113 & 121 & 122 & 123 \\
\end{array}
\]

4. Let $(a)$ be a sequence such that $a_1 = 1$, $a_2 = 1$, and $a_{n+1} = a_n + 2a_{n-1}$ for $n \geq 2$. Prove that $a_n$ is divisible by 3 if and only if $n$ is divisible by 3.

5. Primes and non-primes.
   a) Prove using contradiction that the set of prime numbers is not finite.
   b) For each $n \in \mathbb{N}$, construct a set of $n$ consecutive positive integers that are not prime. (Hint: determine a positive integer $x$ such that $x$ is divisible by 2, $x+1$ is divisible by 3, $x+2$ is divisible by 4, etc.)

6. Let $p$ be a prime number.
   a) Prove that $p$ divides $\binom{p}{k}$ if $1 \leq k \leq p - 1$.
   b) Prove that $n^p - n$ is divisible by $p$ for every $n \in \mathbb{N}$. (Hint: use the binomial theorem and part (a) in a proof by induction.)

PROBLEMS FOR CLASS DISCUSSION

Pair 1 - 5.48, 6.9, 6.15.
Pair 2 - 5.49, 6.7, 6.16.
Pair 3 - 5.50, 6.3, 6.18.