MATH 247 HONORS, FALL 1999 - PROBLEM SET 4

Do five of the following six problems. Due Wednesday, Sep. 22.

1. Consider a balance scale and positive integer weights $w_1 \le \cdots \le w_k$. Let $S_0 = 0$, and let $S_j = \sum_{i=1}^{j} w_i$ for $1 \le j \le k$. Prove that it is possible to balance every integer weight from 1 to S_k if and only if $w_j \le 2S_{j-1} + 1$ for $1 \le j \le k$.

2. *The Game of Nim.* A position in Nim consists of some piles of coins. Two players alternate, with each move removing a portion of one pile. The winner is the player who takes the last coin.

Suppose that the current position has k piles with sizes $n_1, \ldots, n_k$. Let S_j be the set of indices $i \in [k]$ such that the binary representation of n_i has a 1 in position j. For example, when the sizes of piles are $1, 2, 3$, the binary representations are $1, 10, 11$, and we have $S_0 = \{1, 3\}$ and $S_1 = \{2, 3\}$.

Prove that Player 2 has a winning strategy if and only if for every j, the set S_j has even size.

3. Let A be the set of subsets of $[n]$ that have even size, and let B be the set of subsets of $[n]$ that have odd size. Establish a bijection from A to B, thereby proving that $|A| = |B|$. (Such a bijection is suggested below for $n = 3$.)

A	$\emptyset$	$\{2,3\}$	$\{1,3\}$	$\{1,2\}$
B	$\{3\}$	$\{2\}$	$\{1\}$	$\{1,2,3\}$

4. Given real numbers a, b, c, d, let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (ax + by, cx + dy)$. Prove that f is injective if and only if f is surjective.

5. Consider $f\colon A \to B$ and $g\colon B \to A$. Prove that if $f \circ g$ and $g \circ f$ both are identity functions, then f is a bijection. In particular, prove that

a) If $f \circ g$ is the identity function on B, then f is surjective.

b) If $g \circ f$ is the identity function on A, then f is injective.

6. *Iteration.*

a) Suppose that $f\colon A \to B$ is a bijection and that $g\colon B \to B$. Let h be the composition $h = f^{-1} \circ g \circ f$, so $h\colon A \to A$. Derive a formula in terms of f and g for the function from A to A obtained by n successive applications of h.

b) Let $t\colon \mathbb{R} \to \mathbb{R}$ be defined by $t(x) = a(x+b) - b$. Apply part (a) to obtain an explicit formula for the function that is obtained by n successive applications of t.

PROBLEMS FOR CLASS DISCUSION

Pair 1 - 4.11, 4.19, 4.25.

Pair 2 - 4.10, 4.17, 4.28.

Pair 3 - 4.3, 4.13, 4.32.