MATH 247 HONORS, FALL 1999 - PROBLEM SET 2

Do five of the following six problems. Due Friday, Sep. 10, because of Labor Day holiday. Problem set 3 will be distributed Sept. 8 and be due Sept. 15.

1. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$. For the sum and product of f and g (see Definition 1.25), determine which statements below are true. If true, provide a proof; if false, provide a counterexample.

a) If f and g are bounded, then $f+g$ is bounded.
b) If f and g are bounded, then fg is bounded.
c) If $f+g$ is bounded, then f and g are bounded.
d) If fg is bounded, then f and g are bounded.
e) If both $f+g$ and fg are bounded, then f and g are bounded.

2. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$.

a) Prove that f can be expressed in a unique way as the sum of two functions g and h such that $g(-x)=g(x)$ for all $x \in \mathbb{R}$ and $h(-x)=-h(x)$ for all $x \in \mathbb{R}$. (Hint: find a system of linear equations for the unknowns $g(x)$ and $h(x)$ in terms of the known values $f(x)$ and $f(-x)$.)

b) When f is a polynomial, express g and h in terms of the coefficients of f.

3. Given a polynomial p, let A be the sum of the coefficients of the even powers, and let B be the sum of the coefficients of the odd powers. Prove that $A^2-B^2=p(1)p(-1)$.

4. *Extremal problems.*

a) Let f be a real-valued function on S. In order to prove that the minimum value in the image of f is β, two statements must be proved. Express each of these statements using quantifiers.

b) Let T be the set of ordered pairs of positive numbers. Define $f\colon T \to \mathbb{R}$ by $f(x,y)=\max\{x, y, \frac{1}{x}+\frac{1}{y}\}$. Determine the minimum value in the image of f. (Hint: test various pairs to develop a hypothesis about what the minimum is and what pair (x,y) achieves it. Then prove this hypothesis.)

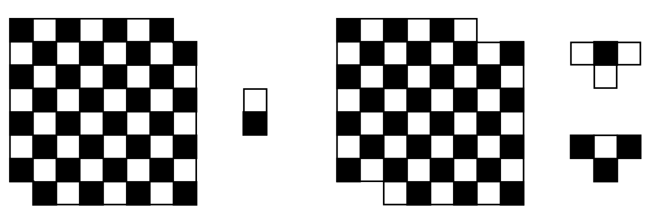
5. For each statement below about natural numbers, decide whether it is true or false, and prove your claim. Use only arithmetic and inequalities and natural numbers, not calculus or graphs of functions.

a) If $n \in \mathbb{N}$ and $n^2+(n+1)^2=(n+2)^2$, then $n=3$.
b) For all $n \in \mathbb{N}$, it is false that $(n-1)^3+n^3=(n+1)^3$.

6. *Checkerboard problems.*

a) Two opposite corner squares are deleted from an eight by eight checkerboard. Prove that the remaining squares cannot be covered exactly by dominoes (rectangles consisting of two adjacent squares).

b) Two squares from each of two opposite corners are deleted as shown on the right below. Prove that the remaining squares cannot be covered exactly by copies of the "T-shape" and its rotations.



PROBLEMS FOR CLASS DISCUSION - CHAPTER 2

All problems during the course will be from the second edition.

Pair 1 - 2.24, 2.32, 2.38.

Pair 2 - 2.14, 2.20, 2.44.

Pair 3 - 2.10, 2.28, 2.51.