MATH 247 HONORS, FALL 1999 - PROBLEM SET 15

WARMUP PROBLEMS: 16.1, 16.12, 16.19, 16.23. Do not write these up. These are easier problems to check understanding of the material.

WRITTEN PROBLEMS. Do five of the following six problems; full credit requires complete justifications in sentences. Final deadline Friday, Dec. 10. However, we have covered all relevant material for this homework by Friday, Dec. 3; homework submitted by Dec. 8 will be graded earlier. Final examination Thursday, Dec. 16, 1:30-4:30PM, 343 Altgeld Hall.

1. Derive the product rule for differentiation using difference quotients. (Hint: Add and subtract an appropriate quantity to the numerator. Use an $\epsilon/2$ argument and the definition of derivative.)

2. Compute the derivative of the cube root function using either definition. (Hint: Use the factorization $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ to simplify the difference of cube roots. Multiply the numerator and denominator by an appropriate constant.)

3. Suppose that $|f(x) - f(y)| \le |g(x) - g(y)|$ for all $x, y \in \mathbb{R}$, and suppose g is differentiable at a with $g'(a) = 0$. Prove using difference quotients that f is differentiable at a and that $f'(a) = 0$.

4. A company wishes to set the price of its new liquid to maximize profit. A marketing analysis indicates that if the price is set at x dollars per gallon, then the number of gallons sold per day will be $g(x) = 1000/(5+x)$. The government, wishing to stimulate production, will also pay the company (per day) \$50 times $\sqrt{g(x)}$. Determine the maximum and minimum values of the company's daily profit and the prices that yield these values.

5. Suppose that $m_1, \ldots, m_k$ are nonnegative real numbers with sum n.

a) Using calculus and induction, prove that $\sum_{i<j} m_i m_j \le (1 - \frac{1}{k})\frac{n^2}{2}$, with equality only when $m_1 = \cdots = m_k$.

b) In the case where $m_1, \ldots, m_k$ are integers, give a combinatorial proof that $\sum_{i<j} m_i m_j$ is maximized when each m_i is $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

(Hint: For part (a), apply the induction hypothesis for each possible value of m_k, then choose the best m_k. For part (b), think of a set counted by $\sum_{i<j} m_i m_j$; how does bringing the arguments closer together without changing $\sum m_i$ affect the size of this set?)

6. Let f be differentiable, with $f'(x) < 1$ for all x. Prove that f has at most one fixed point. (Recall that x is a fixed point of f if $f(x) = x$.) (Hint: Assume that f has two fixed points and use the Mean Value Theorem to obtain a contradiction.)

PROBLEMS FOR CLASS DISCUSSION

Preparation for class means solving your group problems before the class discussion.

Pair 1: 16.34 Pair 2: 16.31 Pair 3: 16.22