MATH 247 HONORS, FALL 1999 - PROBLEM SET 14

WARMUP PROBLEMS: 15.3, 15.7, 15.21, 15.23. Do not write these up. These are easier problems to check understanding of the material.

WRITTEN PROBLEMS. Do five of the following six problems; full credit requires complete justifications in sentences. Due Friday, Dec. 3. Test #3 Wednesday, Dec. 1., 7:30-9:30PM, 141 Altgeld Hall, on Chapters 9, 10, 13, 14. Final deadline for Problem set #15 will be Friday, Dec. 10.

1. *Condensation test.*

a) Let $\langle a \rangle$ be a decreasing sequence of positive numbers. Prove that $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{j=0}^{\infty} 2^j a_{2^j}$ converges. (Hint: Compare the series below.)

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots$$
$$a_1 + a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + a_8 + \cdots$$
$$a_1 + a_1 + a_2 + a_2 + a_3 + a_3 + a_4 + a_4 + \cdots$$

b) For $p \in \mathbb{R}$, prove by part (a) that $\sum_{k=1}^{\infty} k^{-p}$ converges if and only if $p > 1$.

2. *Often discontinuous functions.*

a) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$. Prove that f is discontinuous at every real number.

b) Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = 0$ if $x \in \mathbb{Q}$ and $g(x) = cx$ if $x \notin \mathbb{Q}$, where c is a nonzero real number. Prove that g is continuous at 0 and discontinuous at every other real number.

3. Give two proofs that if $|f(x) - f(a)| \leq c\,|x - a|$ for some positive constant c and all x, then f is continuous at a. One proof should use the ϵ, δ definition, and the other should apply general results about continuity.

4. Let f and g be continuous on $[a, b]$.

a) Prove that if $f(a) > g(a)$ and $f(b) < g(b)$, then $f(c) = g(c)$ for some $c \in [a, b]$.

b) Show by example that if $f(a) = (1/2)g(a)$ and $f(b) = 2g(b)$, then there need not exist $c \in [a, b]$ with $f(c) = g(c)$. Prove that such c must exist if $g(x) \geq 0$ for $x \in [a, b]$.

5. Given a positive real number ϵ, prove that there is a positive real number c (depending on ϵ, but not on x or y) such that $|xy| \leq \epsilon x^2 + cy^2$ for all $x, y \in \mathbb{R}$.

6. Find a counterexample to the following statement: If f is a real-valued function of two variables and all the limits described below exist, then

$$\lim_{y\to 0} \lim_{x\to 0} f(x, y) = \lim_{x\to 0} \lim_{y\to 0} f(x, y).$$

(When taking a limit in one variable, other variables are treated as constants.)

PROBLEMS FOR CLASS DISCUSSION

Preparation for class means solving your group problems before the class discussion.

Pair 1: 15.31 Pair 2: 15.17 Pair 3: 15.26