# MATH 247 HONORS, FALL 1999 - PROBLEM SET 13

WARMUP PROBLEMS: 14.25, 14.26, 14.46. Do not write these up. These are easier problems to check understanding of the material.

WRITTEN PROBLEMS. Do five of the following six problems; full credit requires complete justifications in sentences. Due Monday, Nov. 29. (Problem set 14 will be due Dec. 3.) Test #3 (Wednesday, Dec. 1., 7:30-9:30PM) will cover through this material.

**1.** A $k$-ary expansion is *eventually periodic* if after some initial portion, the remainder is a repeating list of some finite length (this includes terminating expansions, where the repeating list is "0").

a) Prove that the every $k$-ary expansion of a rational number is eventually periodic. (Hint: First prove this for rational numbers of the form $j/s$ with $0 \le j < s$. Then use this and $k$-ary expansions of integers to prove the claim in the general case.)

b) Prove the converse of part (a): if the $k$-ary expansion of $x$ is eventually periodic, then $x$ is rational.

**2.** Let $\langle x \rangle$ be the sequence given by $x_1 = 1$ and $x_{n+1} = 1/(x_1 + \cdots + x_n)$ for $n \ge 1$. Prove that $\langle x \rangle$ converges, and obtain the limit.

**3.** Compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Use this to obtain upper and lower bounds on $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (Hint: Rewrite the summand to obtain a telescoping series. Comment: The exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is $\pi^2/6$.)

**4.** Suppose that $\sum a_n^2$ and $\sum b_n^2$ both converge. Prove that $\sum a_n b_n$ converges. (Hint: Use the AGM Inequality and the comparison test.)

**5.** *Convergence of alternating series.*

"If $\langle a \rangle$ is a sequence whose terms alternate in sign, converge to 0, and satisfy $|a_{k+1}| \le |a_k|$ for all $n$, then the series $\sum_{k=0}^{\infty} a_k$ converges."

Give proofs of the statement above by the two methods below:

a) Show that the partial sums form a Cauchy sequence.

b) Use Proposition 13.18 and the Squeeze Theorem.

**6.** Suppose that $\sum a_k$ converges, that $\sum |a_k|$ diverges, and that $L$ is a real number. Prove that the terms of $\langle a \rangle$ can be reordered to obtain a series that converges to $L$.

## PROBLEMS FOR CLASS DISCUSSION

Preparation for class means solving your group problems before the class discussion.

| Pair 1 | Pair 2 | Pair 3 |
|---|---|---|
| 14.35 | 14.34 | 14.39 |