MATH 247 HONORS, FALL 1999 - PROBLEM SET 1

Do five of the following six problems. Due Wednesday, Sep. 1. Future problem sets will be due mostly on Wednesdays.

If you have trouble with the problems, ask for help! The purpose of the homework is to help students learn to apply the material we study.

In mathematics, "construct", "show", "determine", "obtain", etc., are all words that mean "prove". A complete solution to a problem requires justification, unless it explicitly states that no proof is required. Start writing explanations in sentences now!

1. We have two identical glasses. Glass 1 contains x ounces of wine; glass 2 contains x ounces of water ($x \geq 1$). We remove 1 ounce of wine from glass 1 and add it to glass 2. The wine and water in glass 2 mix uniformly. We now remove 1 ounce of liquid from glass 2 and add it to glass 1. Prove that the amount of water in glass 1 is now the same as the amount of wine in glass 2.

2. *Application of the AGM.*
a) Use Proposition 1.4 to prove that $x(c-x)$ is maximized when $x = c/2$.
b) For $a > 0$, use part (a) to find the value of y maximizing $y(c - ay)$.

3. Let $S = \{(x, y) \in \mathbb{N}^2 \colon (2-x)(2+y) > 2(y-x)\}$. Prove that $S = T$, where $T = \{(1,1), (1,2), (1,3), (2,1), (3,1)\}$.

4. Given $n \in \mathbb{N}$, let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 < a_2 < \cdots < a_n$. Let $(-\infty, a)$ denote the set $\{x \in \mathbb{R} \colon x < a\}$. Express $\{x \in \mathbb{R} \colon (x-a_1)(x-a_2)\cdots(x-a_n) < 0\}$ using the notation for intervals (with justification).

5. Determine the images of the following functions.
a) $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2/(1+x^2)$.
b) $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x/(1+|x|)$.
c) $f \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(a, b) = (a+1)(a+2b)/2$. Explain why this is a function.

6. For S in the domain of a function f, let $f(S) = \{f(x) \colon x \in S\}$. Let C and D be subsets of the domain of f. Find an example where $f(C \cap D)$ does not equal $f(C) \cap f(D)$. Prove that in every instance, one of these two sets must contain the other.

PROBLEMS FOR CLASS DISCUSION - CHAPTER 1

Problems from the SECOND edition, available at UpClose Copies.
Group 1 - 1.11, 1.20.
Group 2 - 1.15, 1.43.
Group 3 - 1.20, 1.46.