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Preface

Combinatorics is now a mature discipline. Although some see it as a maelstrom of isolated problems, it has central themes, techniques, and results that make it a surprisingly coherent subject. Meanwhile, it still rewards its students with endless discovery and delight.

This book introduces the reader to a substantial portion of combinatorics. It is not exhaustive in topics, results, or bibliography. However, it is thorough enough to equip the reader with the tools needed to read or do research in combinatorics or to apply combinatorics in other areas of mathematics and computer science. It assumes the maturity and sophistication of graduate students without assuming prior exposure to combinatorics. It assumes basic undergraduate mathematics, such as elementary set theory, induction, equivalence relations, limits, elementary calculus, and some linear algebra.

More advanced or specialized material is planned to appear in *The Art of Combinatorics*, a four-volume series of texts intended for researchers and for advanced graduate courses in combinatorics. Nevertheless, there is enough here to reward substantial study and investigation.

History and Rationale

Despite its fundamental nature and its explosive growth in recent decades, combinatorics still is not a standard part of mathematics instruction. Curricula (and mathematicians) are slow to change.

Combinatorial ideas appear in courses on elementary discrete mathematics, but such courses can be insubstantial. Serious undergraduate courses in combinatorics are seldom required for math majors. Graduate programs do not require combinatorics. Nevertheless, it is an elegant and valuable subject.

In the mid-1980s, I began to teach graduate courses in combinatorics at the University of Illinois. Excellent books existed for many topics, but every general textbook omitted substantial areas. Gathering material for such a textbook, I succumbed to the overabundance of riches before me. With so much beautiful material in combinatorics, the project grew to become four rotating courses taught from four books, now called *The Art of Combinatorics*.

In 1996, I realized that this structure served only students already committed to focusing on combinatorics. For others, an overview of the subject could have great value. An educated mathematical scientist should know some algebra and analysis, and also such a person should be acquainted with fundamental combinatorics and its relationships to other areas. Furthermore, disparities in preparation of entering students make a core course worthwhile to establish a common background before studying advanced material in combinatorics.

In 1997, I started a one-semester overview course to serve these goals. I extracted the fundamental material from *The Art of Combinatorics* and organized it to emphasize connections among topics. This book is the result. However, with so much beautiful combinatorics to choose from, I could not bring myself to cut the book down to one semester. It can support a two-semester sequence, analogous to fundamental two-semester sequences in classical areas of mathematics. It can also support various one-semester courses, as discussed later.

Since the scope is large, I have also sought to make the book useful as a research reference, rewarding further study after the courses are over. This leads to a fair amount of optional material allowing the reader to probe farther into the subject, plus remarks that provide statements and pointers to further results. Nevertheless, I still aim to keep the material accessible to graduate students.

Organization

One can organize combinatorial mathematics in many ways: by structures discussed, types of questions, methods used, etc. In a broad overview, the connections among topics are as important as the groupings within topics.

Most presentations of elementary combinatorics begin with enumeration or with graph theory; the former is the more classical approach. Natural enumerative questions arise in elementary graph theory, and many graph-theoretic arguments use basic counting techniques, so each informs the other. Here the basic notions of trees, cycles, and isomorphism are stated in Chapter 0 so that enumerative problems about graphs can be used as examples in Part I.

Part I presents the basics of bijective arguments, recurrence relations, generating functions, and inclusion-exclusion, with enhancements. Young tableaux and the elementary aspects of Pólya–Redfield counting appear here from a combinatorial point of view. Deeper algebraic aspects of enumeration are omitted.

Part II pursues central themes of elementary graph theory while reaching important and classical results, particularly those having broad applications. Graph theory is now a huge subject, so selecting fundamental core material is difficult. Many large topics are mentioned here at most in passing or in exercises; these include automorphism groups, Cayley graphs, graph representations, reconstruction, domination, decomposition, packings, genus, minors, nowhere-zero flows, Tutte polynomials, graph labelings, and structured families of graphs.

Part III explores our most general structural object: families of sets, generalizing graphs to hypergraphs. Four aspects of set systems are studied: Ramsey theory, extremal set theory, the structure of partially ordered sets and matroids, and combinatorial designs. Many aspects of posets and enumeration known as algebraic combinatorics are omitted (but Möbius inversion is in Chapter 15).

Part IV develops methods from probability, algebra, and geometry/topology and applies them to questions concerning graphs and sets. Also included are some applications of combinatorics to geometric questions. When discussing methods and connections, it helps to have the terminology and basic results of graph theory and enumeration available. Thus the material in the latter half of the book does depend on the earlier half.

Some topics omitted here are explored in *The Art of Combinatorics* or in my earlier *Introduction to Graph Theory*, which is a more patient and less sophisticated introduction to elementary graph theory. Some important topics in applied discrete mathematics are largely omitted here, partly because they often already have their own well-established courses; prominent among these are coding theory and linear programming.

The nearly 2200 exercises here apply ideas from the text and/or explore further concepts. Many have not appeared in texts before. More than 300 have hints with the problem statements, and another 380 have hints in the back of the book. Solutions are available to instructors via the book's website at

http://www.cambridge.org/west .

I have tried to indicate difficulty by marking easier problems with (−) and harder problems with (+). Problems of intermediate difficulty that are particularly interesting or instructive are marked with (◊). There is much ambiguity (and taste) in these designations, partly because the difficulty of finding a solution is not proportional to its length or its complexity. Thus these labels should be taken lightly.

Usage

Most schools have few regular graduate courses in combinatorics. At such schools this book is appropriate for a two-semester sequence to give a thorough introduction. Instead of separating graph theory from other topics to make two courses, this text integrates the topics into a coherent whole.

This approach enables students from other mathematical areas to acquire the fundamental material about enumeration, graphs, and sets in the first semester without continuing to the second. Also, topics that are best appreciated after knowing the fundamentals in several areas of combinatorics are omitted from courses that study only one part of combinatorics. Examples include the existence (Chapter 14) or construction (Chapter 10) of graphs with large chromatic number and girth, powerful techniques like the Regularity Lemma and entropy (Chapter 11), the application of projective planes to extremal graph problems (Chapter 13), the understanding of optimization via matroids (Chapter 11), and combinatorial applications of linear algebra and topology (Chapters 15 and 16). With the approach here, such applications enliven the second semester.

Nevertheless, the text can also be used separately for courses in graph theory and in "other" combinatorics, as described later.

In a two-semester sequence where most students take the full year and prior combinatorics is not expected, one can focus on Parts I and II in the first semester, III and IV in the second. Parts I and II concisely present the basic material taught in most undergraduate courses on combinatorics and graph theory, with a deeper point of view for graduate students. Students with prior exposure to the subject also benefit from this discussion. Classical topics in graph theory reside in Part II, but interactions between graphs and other topics and techniques appear in other chapters. Later topics are more independent, but the order of presentation here works well. Part III can be viewed as a third introductory area; it considers basic questions about sets and order relations. The methods of Part IV then apply to questions about the combinatorial contexts introduced in Parts I–III.

When the second semester is optional, with the first being the exposure that students from other areas will have to combinatorics, the first semester should be broader. Such a course has goals like those in a one-semester core course leading to multiple advanced courses in combinatorics. With this in mind, I have designated some sections and subsections as "optional", and in others the items marked "*" are optional. Such material is more technical or advanced and can be skipped at first reading without loss of continuity. In such a course, optional or more difficult topics should be skipped in Chapters 1–9 in order to present highlights of early portions of Chapters 10 and 12–14. I used this approach in a one-semester introduction at the University of Illinois that served as a departmental graduate exam course in combinatorics and prepared combinatorics students for four independent advanced courses.

That was a fast-moving course. I spent 16 lectures on enumeration (this can be less), 12 on graphs, and several each on Ramsey theory, posets, probabilistic methods, and designs, aiming in the latter topics mostly to introduce the basic ideas. Thus I was quite brief about signed involutions, the pattern inventory, and the magical properties of Young tableaux. Most students already had some acquaintance with graph theory (the clientele included many computer science students), so from Chapter 5 I presented just a few highlights and left the rest as background reading. In Chapters 6–9 I covered mostly fundamentals. Graduate students should see a bit extra, so in Chapter 6 I proved Plesník's Theorem instead of Petersen's Theorem, in Chapter 7 I gave the lower bound on connectivity of graph products, in Chapter 8 I approached Brooks' Theorem through list coloring (an important theme in modern graph theory), and in Chapter 9 I explained discharging. I skipped the subsections marked optional and also the proof of the Perfect Graph Theorem. In Section 9.3, one can present just enough about discharging to convey the idea.

In Chapter 10, I then presented a few choice pigeonhole applications and the main Ramsey theorem with several applications. In Chapter 12, the main goals were Dilworth's Theorem and LYM Orders for their connections with graph theory. Chapter 14 came before Chapter 13 to provide more time for homework problems, reaching the Local Lemma and threshold functions. In Chapter 13 the goal was the connection between latin squares and projective planes and the application of projective planes to extremal problems in graph theory.

Graph Theory vs. Combinatorics

Many institutions still have separate courses in graph theory and "other" parts of combinatorics, partly due to faculty interests.

A course in enumerative and set-theoretic combinatorics can use Part I and Part III. From graph theory one needs only the definitions from Chapter 0 to present Cayley's Formula (Section 1.3) and count isomorphism classes (Section 4.2). The discussion of chromatic polynomials in Section 4.1 using inclusion-exclusion can be skipped. Part II can be skipped completely. In Part III, one can skip graph-theoretic applications of the Pigeonhole Principle in Section 10.1, graph Ramsey Theory in 10.2, and all of 11.1. The material of Chapters 12 and 13 (except for symmetric chain decomposition of LYM orders and graph-theoretic applications of projective planes) is mostly accessible without graph theory.

I have used Part II for a graph theory course for masters students at Zhejiang Normal University in China. Moving more slowly to accommodate language difference, I did not go much beyond Part II. One would cover Part II except for some optional material. It does not require Part I except for binomial coefficients and simple bijective arguments (counting two ways). In Part III one can use graph-theoretic pigeonhole examples and graph Ramsey theory, possibly adding Section 11.1. The material on matroids in Section 11.3 generalizes results in graph theory, but it takes substantial time to develop the properties. It is more beneficial to include the basic material of Chapter 14, since probabilistic techniques are so effective and important and easily illustrated with graphs.

When a school has separate graduate courses in basic enumeration and basic graph theory, most likely most of the material from Parts III and IV will not be included, depending on the needs of the students and choices by the instructor. A subsequent course requiring the two basic courses can then cover topics from the last two Parts. Although the text often mentions connections between chapters, the chapters after Part II are relatively independent except for the background of language from the early parts.

It is worth noting that a two-semester sequence at U. Nebraska has for about 10 years used Part I combined with supplementary material on coding and information theory in the first semester, and Part II combined with selections from Chapters 10–15 in the second semester.

Highlights

Indeed, the connections between topics are among the features of this book. One aspect is pedagogical: we solve several fundamental problems repeatedly to show the usefulness of various techniques.

For example, Cayley's Formula to count labeled trees is obtained bijectively in Chapter 1, inductively in exercises, via generating functions and Lagrange Inversion in Chapter 3, and in Chapter 15 via the Matrix Tree Theorem and via eigenvalues. Derangements are counted via recurrence, generating functions, and inclusion-exclusion; Catalan numbers are also obtained repeatedly, including from Young tableaux. Turán's Theorem on the maximum number of edges in a graph containing no $(r + 1)$-vertex complete graph is proved inductively in Chapter 5, via extremality in Chapter 11, algebraically in exercises in Chapter 11, and probabilistically in Chapter 14. Planar graphs are characterized inductively in Chapter 9, via matroids in Chapter 11, and via dimension of partial orders in Chapter 16. The König–Egerváry max/min relation of Chapter 6 involving matchings and vertex covers in bipartite graphs is shown to be equivalent to Dilworth's Theorem on posets in Chapter 12 and is a special case of the Matroid Intersection Theorem in Chapter 11.

Other connections arise when techniques from one context are used to solve problems from other contexts. For example, common systems of distinct representatives (characterized in Chapter 7 using Menger's Theorem) are used to obtain symmetric chain decompositions of LYM orders in Chapter 12. Extremal problems for diameter in Chapter 5 and for graphs without 4-cycles in Chapter 11 are attacked using projective planes in Chapter 13. The problem in Chapter 8 of finding small triangle-free graphs with large chromatic number is discussed using Ramsey theory in Chapter 10. Probabilistic methods from Chapter 14 are used to obtained good bounds on crossing numbers in Chapter 16. Bounds on the graph connectivity parameter from Chapter 7 are obtained via eigenvalues in Chapter 15. Bounds on the list chromatic number, introduced for a richer study of graph coloring in Chapter 8, are obtained using the Discharging Method in Chapter 9 and the Combinatorial Nullstellensatz in Chapter 15. Pym's Theorem from Chapter 7 is used to prove the Planar Separator Theorem in Chapter 9.

Finally, since this is a graduate textbook, it covers the standard material of an elementary introduction efficiently in order to go beyond and offer the reader more. This permits the inclusion of many jewels that a standard elementary introduction at the undergraduate level cannot reach. Some of these were mentioned earlier in describing my overview course. Here are more of them.

In enumeration, we become familiar with the Delannoy numbers and the Eulerian numbers as additional basic counting models. We explain techniques such as Wilf's Snake Oil technique for evaluating sums, the Exponential Formula for obtaining generating functions, and Lagrange Inversion for extracting coefficients. We explore the combinatorial aspects of Young tableaux, obtaining not only the Hook-Length Formula and the Robinson–Schensted–Knuth Correspondence, but also Greene's Theorem about the largest union of k increasing subsequences in a permutation.

In graph theory, Chapter 5 provides unusual applications of the number of vertices of odd degree being even. Orientations with small outdegree, a beautiful application of Hall's Theorem by Hakimi in Chapter 6, are later applied to list coloring in Chapter 15. Optional material for advanced courses includes the proof of Tutte's f-Factor Theorem and the fast matching algorithm of Hopcroft and Karp in Chapter 6, and in Chapter 7 the Nash-Williams Orientation Theorem characterizing k-connected orientations (generalizing Robbins' Theorem on strong orientations). Also in Chapter 7, the standard sufficient conditions for spanning cycles in graphs are studied extended to long-cycle versions. List coloring provides a modern approach to coloring in Chapter 8, and the full form of Vizing's Theorem for multigraphs is proved. Chapter 9 includes a thorough introduction to the Discharging Method and an accessible proof of the Planar Separator Theorem.

In Part III on sets, we present many beautiful results that are not easy to find in general textbooks. Again some will typically be options for further reading. Chapter 10 includes the proof of the Stanley–Wilf Conjecture on pattern-avoiding permutations and the construction of graphs with large girth and chromatic number. The application of Ramsey's Theorem to table storage in computer science is unusual and makes use of allowing many colors. In Chapter 11, the Regularity Lemma of Szemerédi is presented, applied, and proved. Also, the notion of discrete entropy is developed to obtain extremal bounds on set counting problems. The basic structural aspects of posets are given in Chapter 12, along with important results about poset dimension, and the treatment of lattices leads to a rigorous discussion of correlational inequalities. Beyond the basic results about designs and projective planes, Chapter 13 includes the Multiplier Theorem for difference sets and the disproof of the Euler Conjecture.

Part IV on methods provides tools to attack many problems. Chapter 14 on the Probabilistic Method has a scope similar to the popular textbook of Alon and Spencer, including the basic methods, Dependent Random Choice, the Local Lemma, threshold functions, and concentration inequalities. The scope of Chapter 15 is similar to the well-known notes of Babai and Frankl on *Linear Algebra Methods in Combinatorics*; we also include Kastelyn's use of permanents to count perfect matchings in planar graphs and a discussion of Möbius inversion on posets. The Combinatorial Nullstellensatz is pursued as far as the recent strengthening of Thomassen's famous 5-choosability of planar graphs by Grytczuk and Zhu, show that every planar graph contains a matching whose edge-deletion yields a 4-choosable subgraph. Chapter 16 discusses geometric embeddings of graphs, applications of the Borsuk–Ulam Theorem and its relatives from combinatorial topology, and geometric aspects of partially ordered sets.

I hope this brief sampling whets the appetite for the delights ahead.

Acknowledgments

When C.L. Liu heard in the mid-1980s that I was accumulating text material on combinatorics, he showed me the lecture notes he had published as *Topics in Combinatorial Mathematics* (Math. Assoc. of America, 1972). These came from a summer seminar at Williams College in 1972 and were used in the combinatorics graduate course at the University of Illinois that I inherited from him. He proposed that we work them into a polished textbook; thus began *The Art of Combinatorics*. As described earlier, that project grew beyond the confines of a single volume, and the present text is closer to what he had in mind (but still more than twice as big). I thank him for the suggestion that started the process.

Also worthy of mention is Liu's earlier book *Introduction to Combinatorial Mathematics* (McGraw-Hill, 1968), which in 1972 introduced me to combinatorics. This book established the overall shape and subject matter for modern courses in combinatorics. Before it (at least in the U.S.) there was not much more than a compilation of chapters from eminent researchers who delivered a short course for engineers at UCLA (*Applied Combinatorial Mathematics*, 1964). Courses in elementary graph theory were initially shaped by the seminal textbooks of Berge (1962), Harary (1969), and Bondy and Murty (1976).

This text has benefitted by comments from many users and reviewers. Those who used pre-publication versions of the text several times include Garth Isaak (Lehigh U.), Art Benjamin (Harvey Mudd), Stephen Hartke, Christine Kelley, Xavier Pérez-Giménez, and Jamie Radcliffe (Nebraska), Jozsef Balogh (Illinois), and Mark Kayll and Cory Palmer (Montana). John Ganci and Leen Droogendijk each gave the book an extremely thorough reading, catching many glitches.

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At Cambridge University Press I thank my editor David Tranah for his patience through years of delays as I slowly refined the text. He accurately concluded that I view the book as a "work of art", which is part of why it took so long (25 or 35 years, depending on how you count). Clare Dennison shepherded the book through production, and Sarah Routledge gave it an incredibly thorough proofreading. All remaining errors, which I am sure exist, are solely my responsibility, especially since I continued to squeeze in exercises and make other refinements for months after she finished her job.

This book has been typeset using TEX. The scientific community owes a vast debt to its creator, Donald E. Knuth. With brilliance, foresight, and generosity, he has provided a common language for the publication and communication of technical material that is now used all over the world. Besides its versatility and free availability, its incredible genius is that it runs amazingly fast.

Chris Hartman taught me `perl`, which I used to convert earlier `groff` files to TEX. The "millenial" fonts were developed by Stephen Hartke, who helpfully made it possible to use them with plain TEX instead of LATEX so that I could have greater control over spacing and placement of material. The references and indexes were assembled through herculean effort by Thomas Mahoney, who wrote scripts to handle most of the processing, built an effective computing environment for me to use in both China and the U.S., and patiently helped me resolve all my internet difficulties. Finally, I thank my wife, Ching Muyot, for her assistance, patience, and understanding with the crunch of each year's edition, no matter how many times I declared the book "essentially finished".

Feedback

I eagerly welcome comments on all aspects of this book. This includes selection and presentation of topics, errors made in mathematics or attribution or typography, items missing from the index, suggestions of additional hints, material that should be added if there is ever another edition, etc. Please send comments to dwest@illinois.edu. Errata will be listed at

http://www.math.uiuc.edu/~west/coerr.html .

Enjoy!

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