Resonances in a continuously forced anharmonic oscillator

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Abstract

Periodically forced motion of a classical particle in a one-dimensional potential with superquadratic growth at infinity is considered. It is shown that an arbitrary amount of energy can be transmitted to the oscillator by exciting the system with a continuous time-periodic forcing. This result extends Littlewood's example of unbounded motions in the presence of a discontinuous periodic forcing and, thus, sheds light on the relation between the smoothness of forcing and the stability of motion. A new version of the averaging procedure, which had to be applied to justify the construction, is outlined.

It is a well-known fact that a harmonic oscillator can be resonantly excited with a time-periodic forcing so that its amplitude grows indefinitely with time. With the aim to understand the nature of nonlinear resonances Littlewood asked whether this statement is no longer true for an anharmonic oscillator whose restoring force is stronger than linear. Formally this question is equivalent to the problem of boundedness of solutions of the second order ODE

\[ \ddot{x} + U'(x) = p(t), \] (1)

where \( p(t + 1) = p(t) \) and \( U'(x)/x \to \infty \) as \( x \to \pm \infty \). Intuitively, one might expect that since in such systems the frequency tends to infinity with the amplitude, any resonances would be destroyed and all solutions would stay bounded for all time. However, this intuition turns out to be wrong as Littlewood showed by constructing an oscillator with a superlinear restoring force such that unbounded motions could be excited with a discontinuous periodic forcing of square wave type, see Ref. [4] or [5].

Subsequently, many stability results for different classes of superquadratic potentials were obtained based on KAM theory, see Refs. [2,3,6–8]. It was also found which necessary condition for application of KAM theory is violated in Littlewood's construction. Translated into physical terms the condition requires monotone dependence of the frequency on the amplitude in the unperturbed system, see Ref. [6]. However, no results were obtained on the relation between the smoothness of periodic forcing and the stability of motion.

In this Letter it is shown that an excitation of unbounded motions in superquadratic potentials can be achieved with a continuous periodic forcing. As in Littlewood's example this is done by constructing such a potential that at least one unbounded solution can be found in the presence of a certain periodic force.

Before describing our example, we briefly recall Littlewood's construction, see Fig. 1. We start with the equation

\[ \ddot{x} + 4x^3 = (-1)^{[t]+1}, \] (2)

where \([t]\) is an integer part of \( t \). Next the initial con-
Fig. 1. Resonance conditions and the modification of the potential. Each arrow represents oscillation of the particle during an autonomous phase. If the graph of $U_0 = x^4$ is replaced by $U_0 \pm x$, then the arrows marked $p = \pm 1$ become horizontal.

Condition $(x(0), \dot{x}(0)) = (x_0, 0)$ $(x_0 > 0)$ is chosen so that the solution $x(t)$ would swing left and stop at $x(1) = -x_1$ $(x_1 > 0)$ with $\dot{x}(1) = 0$. If the potential is left unmodified, the solution will make more than one full swing during the second half-period stopping at $-x_2$ $(x_2 > x_1)$ at least once. The potential can be modified in the open interval $(x_0, x_2)$ in such a way that the solution is slowed down so that the point $x_2$ is reached exactly at the end of the period. A simple argument based on the superquadratic character of $x^4$ shows that during the next half-period the solution will again make at least one trip from $x_2$ to $-x_3$, where $x_3 > x_2$. By modifying the potential on the interval $(-x_3, -x_1)$ we slow the solution down so that $x(3) = -x_3$, thus again achieving the resonance condition. Continuing this procedure we obtain an unbounded solution. However, the above argument does not show whether superquadraticity of $U$ is preserved which turns out to be the most difficult part of the construction. It is shown in Ref. [5] that the modification producing an unbounded solution can be such that $|U'(x)| > cx^2$ and that $U$ can be smoothed out without affecting the construction.

In our construction we also start with the equation

$$\ddot{x} + U_0'(x) = p(t),$$

where $U_0(x) = x^4$, $p$ is continuous, periodic of period 4, $p(t + 2) = -p(t)$ and on $t \in [0, 2]$ see Fig. 2. It is known that this equation has no unbounded solution, see Ref. [7]. The goal is to modify the potential so as to obtain at least one unbounded solution $x(t)$ for the new equation.

The idea is to use rapid changes of $p$ at the beginning of each forcing quarter-period to accelerate the solution and then modify the potential so as to bring the system in resonance with the external forcing at the beginning of the next forcing quarter-period.

The main problem in carrying out this program consists in estimating the evolution of the action-angle variables during the forcing period. It is necessary for the following reasons: first, to ensure that the energy gained at the beginning of the forcing quarter-period is not lost in the remaining time, second, to show that by modifying the potential on some interval the solution may be brought in resonance with the external forcing. For the system with quartic potential the averaging procedure can be easily justified by applying the adiabatic invariance theorem, see e.g. Ref. [1]. However, the theorem does not apply to the modified potentials since their derivatives change too rapidly. We will show later how this difficulty can be avoided by using a different averaging procedure.

First, the transformation to the action-angle variables has to be carried out, see Fig. 3. The Hamiltonian of the problem is given by

$$H = \frac{1}{2} \dot{x}^2 + U(x) - p(t) x.$$
Fig. 3. The definition of the action-angle variables. To each point $(x, \dot{x})$ the action $I$ is assigned given by the area enclosed by the level curve through $(x, \dot{x})$ and the angle $\phi$ which is given by the proportion of the area of an infinitesimal ring between the $\dot{x}$-axis and $(x, \dot{x})$. The center of gravity of the infinitesimal ring is at $(x)$. 

$(0, \dot{x}_0 > 0)$ to $(x, \dot{x})$. Thus, the angular variable is defined modulo 1 and therefore the period $T(I, p)$ is reciprocal to the frequency $\omega(I, p)$.

In the action-angle variables the Hamiltonian takes the form

$$K(I, \phi, p) = H_0(I, p) + \dot{p}H_1(I, \phi, p),$$

where $H_0$ is the old Hamiltonian expressed in the new variables and $H_1$ is the partial derivative with respect to time of the generating function producing the above transformation, see Ref. [6] for details. The equations of motion are given by

$$\dot{I} = -\dot{p} \frac{\partial H_1}{\partial \phi}(I, \phi, p),$$

$$\dot{\phi} = \omega(I, p) + \ddot{p} \frac{\partial H_1}{\partial I}(I, \phi, p),$$

(5)

where

$$\frac{\partial H_1}{\partial \phi}(I, \phi, p) = T(I, p) [x(I, \phi, p) - \langle x \rangle(I, p)],$$

see Ref. [9] for the derivation of the last formula.

Now, we give a geometrical explanation of the action increase at the beginning of the forcing quarter-period. Consider the tube in the $(x, \dot{x}, t)$-space whose $t$-cross sections have a constant area $I_0$:

$$T_{I_0} = \{(x, \dot{x}, t) : H(x, \dot{x}, p(t)) = H(I_0, p(t))\}.$$

While $p$ = constant = 1 the trajectory spirals on the tube, see Fig. 4. At $t = 0$ the tube is sheared to the left. If the solution at $t = 0$ finds itself to the right of $x = \langle x \rangle$ then the distance to the tube increases and so does the action (note that by (5) $\dot{I} > 0$ if $x > \langle x \rangle$). The action decreases when the solution enters $x \leq \langle x \rangle$. However, this decrease is smaller than the preceding increase. Indeed, if at $t = 0$ (when $\dot{p} = \infty$) the solution is at $x = \langle x \rangle$ then it spends the largest possible time in the region where $x \geq \langle x \rangle$ and by the time it enters the region where $x \leq \langle x \rangle$, $\dot{p}$ gets smaller making the change in $I$ smaller. The result is that more action is gained than lost after each revolution while $\dot{p}$ is sufficiently large. This effect is due to the monotonicity of $\dot{p}$.

Now, we will show how the above mechanism of action increase is used to create an unbounded solution. The procedure is based on the repeated use of the resonance condition and is similar to Littlewood’s construction.

We define the instantaneous right and left turning points $\pm x_\pm = \pm x_\pm(x, \dot{x}, t)$ associated with a point $(x, \dot{x}, p(t))$, see Fig. 3, as the solutions of the equation

$$H(\pm x_\pm, 0, p(t)) = H(x, \dot{x}, p(t)).$$

Thus, $-x_-$ and $x_+$ are the points where the solution would turn around if the system were frozen at the moment $t$. Our modification of $U_0 = x^4$ leaves $U_0$ unchanged at a sequence of points $\{(\pm 1)^k A_k\}$, where $A_{k+1} > A_k > 0$ and $(-1)^k A_k$ is the point where the solution turns around on the autonomous intervals $t \in [2k - 1, 2k]$. In contrast with Littlewood’s construction, the sequence $A_k$ cannot be defined by a recursive formula, although it is constructed so that it increases.
according to the asymptotic relation$^1$, see Ref. [9] for details,

\[ A_{n+1} - A_n \sim \frac{1}{A_n^2 + \epsilon}. \]  

(6)

It is important for our construction that such sequence is unbounded (indeed $A_n \to \infty$ leads to an immediate contradiction $0 = 1/A$).

We describe the inductive procedure which produces an unbounded solution.

Let the initial condition $(x(0), \dot{x}(0))$ be chosen so that it satisfies the resonance condition $x = (\dot{x})$ and $\dot{x} > 0$, with a yet unspecified $I_0$. Had the system remained autonomous, the solution would oscillate between $x_+(0) = A_0$ and $-x_-(0) = -(A_0)_-$, where $-(A_0)_-$ is defined by the energy relation. But in our nonautonomous system the solution moves as shown in the diagram: the turning points drift to the left as $p$ decreases from 1 to $-1$. At $t = 1$ the system becomes autonomous again, and the solution oscillates between $A_1$ and $(A_1)_+$. Because of the resonance condition the action has increased, $A_1 > A_0$, see Ref. [9] for the estimates (6). To repeat this increase $A_2 > A_1$ we need to fulfill the resonance condition at $t = 2$. This is done by increasing $I_0$ (or, equivalently, $A_0$). The number of full rotations of $(x, \dot{x})$ in the phase plane during $t \in [0, 2]$ thus increases, and therefore for some $A_0$ the resonance condition is satisfied. To create the resonance again at $t = 4$ the potential is deformed in the upward direction, see Fig. 5, on $(A_0, A_2)$ in a special way with a parameter $\nu$ ($\nu_1 \leq \nu \leq \nu_2$), see Ref. [9] for the rigorous construction of the modification. As a solution passes through the deformed part of the potential, it slows down compared to the original solution. If for some deformation of the potential the number of full rotations on $t \in [2, 4]$ decreases by 2, then the solution vector $z^\nu(t)$ makes a full revolution in $(x, \dot{x})$ as $\nu$ travels through $[\nu_1, \nu_2]$. Therefore for some intermediate $\nu$-value the resonance condition at $t = 4$ is satisfied. Repeating this procedure each half-period we obtain an unbounded solution.

As we have already observed, in order to justify both steps of the construction, outlined above, we have to estimate the evolution of the action variable in the class of “wavy” potentials. This requires, in particular, an adiabatic invariance theorem. Unfortunately, none of the standard adiabatic invariance results (see e.g. Ref. [11]) apply to our system since it is not $C^1$-close to an integrable one. Nevertheless, the following adiabatic invariance result with weak assumptions on $U$ holds.

If the potential $U(x)$ is $O(\sqrt{|x|})$ close to $U_0(x) = x^4$, $U'(x)$ grows not slower than $|x|^{3/2}$, and $|||p|||_1 \geq 2$ is bounded, then the action variable is an adiabatic invariant, i.e.

\[ I(1) - I(0) = O(1/\omega(0)). \]

Usually the adiabatic invariance is established by changing the variables so as to move $(\phi, t)$-dependence to higher order terms. However, such an approach requires estimates on the growth rate of $|H|_2$. These estimates hold for the well-behaved potentials, e.g. $U_0 = x^4$, but unfortunately they do not hold for “wavy” potentials employed in the construction.

Below, we show how the above approach can be modified so that boundedness on the growth rate of the derivatives will be no longer required. Instead, boundedness on the growth rate of some finite differences will have to be assumed. These finite differences can be estimated directly.

Since $||p||_1$ is bounded, the equations of motion can be rewritten in the following form,

\[ \dot{I} = -f(I, \phi, p), \]

\[ ||f||_k = \text{sup} \{f, f', \ldots, f^{(k)} \}. \]

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$^1$ In this paper “$a_0 \sim b_n$” means $0 < c \leq a_0 / b_n \leq C$, where $c$ and $C$ are independent of $n$.
\[ \dot{\phi} = \omega(I, p) + g(I, \phi, p), \]

where \( f \) and \( g \) depend smoothly on \( p \). In this system \( 1/\omega(0) \) can be used as a small parameter since \( \omega \to \infty \) as \( I \to \infty \).

For convenience we give a sketch of the standard procedure which establishes the adiabatic invariance, see e.g. Ref. [1], in order to indicate the step which fails for the "wavy" potentials; then we present the modified averaging procedure that works.

Usually one looks for the change of variables in the following form, \( J = I + (1/\omega_0) P(I, \phi, p) \), then the equation for the rate of change of \( J \) is given by

\[
j = \dot{I} + \frac{\omega}{\omega_0} \frac{\partial P}{\partial \phi} + O\left( \frac{1}{\omega_0} \right)
= -f(I, \phi, p) + \frac{\omega}{\omega_0} \frac{\partial P}{\partial \phi} + O\left( \frac{1}{\omega_0} \right),
\]

provided \( ||P||_1 \) is bounded.

Since \( \langle f \rangle = 0 \), the function

\[ P = \frac{\omega}{\omega_0} \int_0^\infty f(I, \psi, p) \, d\psi, \]

is periodic in \( \phi \). Then under some assumptions on the growth rate of \( f, g, \omega \) and their derivatives, \( ||P||_1 \) is bounded and we obtain

\[ J = O(1/\omega_0), \]
\[ J - I = O(1/\omega_0), \]

and therefore \( I(1) - I(0) = O(1/\omega_0) \).

From the above observations it is easy to see that \( ||P||_1 \) is not bounded for our "wavy" potential since \( f, g, \omega \) do not behave well.

We show now how the above averaging procedure can be adjusted to work for the modified potentials by making the following steps:

1. Between two consecutive intersections at \( t = t_n \) and at \( t = t_{n+1} \) of the solution with the section \( \phi = 0 \) we approximate the evolution of the action in the original system by that in the corresponding autonomous one,

\[
I(t_{n+1}) - I(t_n) = \int_{t_n}^{t_{n+1}} f(I, \phi, p(t)) \, dt
= \int_{t_n}^{t_{n+1}} f(I_n, \omega(I_n, p_n)(t-t_n), p_n) \, dt
+ \int_{t_n}^{t_{n+1}} [f(I, \phi, p(t))
- f(I_n, \omega(I_n, p_n)(t-t_n), p_n)] \, dt.
\]

(2) Taking the sum over all revolutions during \( t \in [0, 1] \) and suppressing the dependence of \( f \) on \( I, \phi, p, \) we obtain

\[
I(1) - I(0) = \sum_{n=0}^N \int_{t_n}^{t_{n+1}} f_n(t) \, dt
+ \sum_{n=0}^N \int_{t_n}^{t_{n+1}} [f(t) - f_n(t)] \, dt,
\]

where \( N \sim \omega_0 \). The integrals in the first sum would vanish if they were taken over the intervals \( t \in [t_n, t_n + \omega^{-1}(I_n, p_n)] = [t_n, t_n + T_n] \), because \( \langle f \rangle = 0 \).

It is assumed that \( t_0 = 0 \) and \( t_N = 1 \), however, there is no loss of generality in these assumptions since \( p(t) \) can be taken constant outside the interval \( (0, 1) \) without affecting \( I(1) - I(0) \).

(3) Thus, the change of action is \( O(1/\omega_0) \) if \( T_n - (t_{n+1} - t_n) = O(1/\omega_0^2) \) and \( f - f_n = O(1/\omega_0) \) for \( t \in [t_n, t_{n+1}] \) for all \( n \leq N \).

Therefore the problem is reduced to estimation of finite differences of some functions on the solutions of autonomous and nonautonomous systems.

Now, we show how the differences can be estimated directly even if the corresponding derivatives are unbounded. As an example we consider estimation of the difference of periods.

Let \( U(x) \) be a modified quartic potential satisfying the estimates in the statement on adiabatic invariance, then the following result holds:

\[ \text{If } H_1 - H_2 = O(\sqrt{x_4(H_1)}), \text{ then } T(H_1) - T(H_2) = O(1/x_4^2(H_1)), \]

where \( T(H) \) is the period of the

\[ \text{3 The condition } H_1 - H_2 = O(\sqrt{x_4(H_1)}) \text{ comes from estimates on how much energy the solution can lose or gain during one revolution in the phase plane.} \]
solution of the autonomous system $\ddot{x} + U'(x) = 0$ with energy $H$ \footnote{We will write $x_+$ instead of $x_+(H_1)$ because $x_+(H_1) \sim x_+(H_2)$.}

Note that $\Delta T = T(H_1) - T(H_2)$ cannot be estimated by using bounds on the derivatives for the reasons explained above. The following formula was obtained by Levi in Ref. \cite{6} for $T'$,

$$T'(H) = \frac{\sqrt{2}}{2H} \frac{x_+(H)}{1 - \frac{W''}{(V')^2}} \frac{dx}{\sqrt{H - V(x)}}.$$  \hspace{1cm} (7)

For the quartic potential this formula gives a good estimate $T'(H) \leq C(T/H) \approx 1/x_+^2$ since $W''/(V')^2$ is bounded, but for deformed potentials $T'(H)$ may not only fail this bound but it can take on even infinite values at some points. More precise estimates show that $T'(H)$ indeed exceeds the required bound for some values of $H$ if the quartic potential is modified as in our construction.

Here, we estimate $\Delta T$ directly, see Fig. 6,

$$T(H_1) - T(H_2) = \int_{x_+(H_1)}^{x+(H_2)} \frac{dx}{v_1(x)} - \int_{x_-(H_1)}^{x_+(H_2)} \frac{dx}{v_2(x)} = 2\sqrt{2} \int_{-x_-(H_1)}^{x+(H_2)} \frac{dx}{\sqrt{H_1 - U(x)}} - 2\sqrt{2} \int_{-x_-(H_2)}^{x+(H_2)} \frac{dx}{\sqrt{H_2 - U(x)}}.$$ \hspace{1cm} (8)

For convenience we introduce a cut-off function $\gamma(x)$ satisfying

1. $\gamma(x) \in C^\infty(R)$,
2. $0 \leq \gamma(x) \leq 1$,
3. $\gamma(x) = 0$, if $|x| \leq 1$ and $\gamma(x) = 1$, if $|x| \geq 2$.

Using this inequality $x^4 \leq U(x) \leq x^4 + \gamma(x) C \sqrt{|x|}$, the difference of the integrands on the interval where $x^4 + \gamma(x) C \sqrt{|x|} \leq \min(H_1, H_2)$, can be majorized by substituting $x^4 + \gamma(x) C \sqrt{|x|}$ instead of $U(x)$. Indeed, this substitution decreases the expressions under the square root signs by the same value; the difference of the fractions, therefore, increases (this follows from the monotonicity of the derivative of $1/\sqrt{x}$). The difference of the integrals over this “middle interval” can now be estimated by (7). The remaining left and right intervals have lengths of order $x_+^{-2.5}$, therefore they are passed in time less than $C \frac{x_+^{-2.5}}{x_+^{1.5}} = C/x_+^2$. Thus we obtain that $\Delta T = O(1/x_+^2)$ provided $\Delta H \leq C \sqrt{x_+}$, as desired.

In summary, it has been shown that continuously forced oscillations of a classical particle in a superquadratic potential can be unbounded. The example of instability has been presented with the averaging procedure making the construction possible. We would like to note that the averaging technique outlined in the Letter is applicable to a larger class of systems including, in particular, adiabatically forced Hamiltonian Systems with one degree of freedom governed by the equation

$$\ddot{x} + \left[ v_0(x) + \epsilon V_1 \left( \frac{x}{\epsilon} \right) \right]' = p(\epsilon t).$$

Using the above averaging technique one can show that this system possesses an adiabatic invariant under some conditions on $V$ and $p$.

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