π-Kinks in the parametrically driven sine-Gordon equation
and applications

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Abstract

Parametrically driven sine-Gordon equation with a mean-zero forcing is considered. It is shown that the system is well
approximated by the double sine-Gordon equation using the normal form technique. The reduced equation possesses π-kink
solutions, which are also observed numerically in the original system. This result is applied to domain walls dynamics in
one-dimensional easy-plane ferromagnets. For such system the existence of π-kinks implies the “true” domain structure in
the presence of high-frequency magnetic field. Copyright © 1998 Elsevier Science B.V.

Keywords: Sine-Gordon equation; Parametrical forcing; Domain wall dynamics; Easy-plane ferromagnets; Normal form technique

1. Introduction

The sine-Gordon equation (SGE) is a frequent object of study in numerous physical applications, such as Josephson
junction transmission lines [1,2], dislocations in crystals [3], charge density waves [4], waves in quasi-one-
dimensional ferromagnetic materials [5–8]. Ordinary sine-Gordon equation for a scalar phase field \( \phi \) supports stable
traveling wave solutions (solitons), with boundary conditions \( \phi = 0 \) and \( \phi = 2\pi \), called 2π-kinks. These solutions
represent localized moving transition zones separating regions of constant phase \( \phi \), where the phase has the same
value in both regions 0, 2\( \pi \). The only homogeneous solution of SGE, other than \( \phi = 0 \), is \( \phi = \pi \), which is unstable.
Therefore, 2π-kinks are the only traveling wave solutions of SGE. However, kinks separating regions with different
values of the phase are very important since, unlike 2π-kinks, their spatial field average has a time dependence. To ob-
tain such kinks the first step is to modify SGE so that it possess another stable equilibrium \( \phi = \) constant, e.g. as it was
done in [9]. In that paper the authors obtained a 2π-kink consisting of two connected π-kinks: 0 \( \rightarrow \) π and π \( \rightarrow \) 2\( \pi \).

Significant progress has been achieved in analytical and numerical studies of soliton dynamics in quasi-one-
dimensional ferromagnets with strong anisotropy of a hard magnetization axis [5–7,10,11]. These solitons were
observed in various experiments [12,13]. The anisotropy keeps spins close to an easy plane with parallel magneti-
zation. If a constant external magnetic field is applied within the easy plane then, under some conditions, the spin

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PII S0167-2789(98)00129-8
dynamics can be described by SGE (see, e.g., [6]). Then the only localized traveling wave solutions are $2\pi$-kinks which correspond to moving walls between adjacent domains with parallel magnetization. Motion of a $2\pi$-kink preserves the average magnetization. However, it is very important to obtain such an interesting phenomenon as evolution of the average magnetization on the long time scale. This occurs when adjacent domains have anti-parallel magnetization which corresponds to $\pi$-kinks or "true" domain walls. Such $\pi$-kinks have been obtained by introducing an additional anisotropy within the easy plane [6,14].

In this paper we show that a parametrically excited SGE with a fast mean-zero forcing possesses $\pi$-kink solutions. This can be applied to easy-plane anisotropic ferromagnets, with $\pi$-kinks corresponding to moving true domain walls. The parametric excitation is then realized by a rapidly oscillating external magnetic field. We require no additional anisotropy of the ferromagnet for the existence of $\pi$-kinks.

Although a parametrically forced SGE has been considered in the context of Josephson junction [1,2], waves in ferromagnets [11], and other applications [9,15], the mean-zero case has not been studied. However, this special case deserves detailed consideration as it corresponds to qualitatively new behavior of solitary waves. As we will show later, a mean-zero periodic excitation implies the existence of $\pi$-kinks for which the field spatial average evolves in time.

The Hamiltonian of the ferromagnet with an easy-plane anisotropy in the presence of an external magnetic field is given by [6,16]

$$\mathcal{H} = \sum_{i=1}^{N} \left[ -JS_i \cdot S_{i+1} + D(S_i^z)^2 - g\mu_B H S_i^x \right],$$

where $S_i$ are dimensionless classical spin vectors, $N$ is the number of spins, and $J$ is a constant of exchange interaction. The external magnetic field $\mathbf{H}$ is directed along the $X$-axis. The anisotropy constant $D$ provides the existence of an easy plane, $XY$, at each site of the chain.

The energy of interaction of the spins with the applied field is assumed to be much weaker than that of the anisotropy:

$$DS \gg g\mu_B H_0,$$

where $H_0$ is the amplitude of the field. In the spherical coordinates the spins in Eq. (1) are given by

$$S_i = S(\cos \theta_i \cos \phi_i, \cos \theta_i \sin \phi_i, \sin \theta_i).$$

The dynamics of spin $S_i$ is given by [14]

$$\hbar \dot{S}_i = S_i \times \frac{\partial \mathcal{H}}{\partial S_i}.$$  

Without loss of generality, the chain of spins can be oriented along the $X$-axis. We can, therefore, come to a continuous description, for a small distance between neighboring spins, $a_0$:

$$\phi_x(x) \sim \frac{\phi_{i+1} - \phi_i}{a_0}. $$

Using conditions (2) and (5) we arrive, after some transformations, at

$$\phi_{xx} - \frac{\phi}{C^2} = vH \sin \phi,$$

where

$$C = \frac{2a_0 S^2 \sqrt{JD}}{\hbar}, \quad v = \frac{g\mu_B}{2JSa_0^2}. $$
In this paper we consider a rapidly oscillating external field $H = H_0 \alpha(t/\varepsilon)$, where $\alpha$ is a mean-zero periodic function with a unit amplitude. As we will show below, this choice of the field provides the existence of $\pi$-kinks. Rescaling time and coordinate to dimensionless variables, $\tilde{t} = t \sqrt{\varepsilon H_0}$, $\tilde{x} = x \sqrt{\varepsilon H_0}$, we obtain (after dropping tildes) a parametrically forced SGE

$$\phi_{tt} - \phi_{xx} + a(t/\varepsilon) \sin \phi = 0.$$  \hspace{1cm} (8)

If Eq. (8) were averaged formally over the fast time scale then a linear wave equation would be obtained. This linear equation has the wave propagation velocity $c = \pm 1$. Indeed, the numerical simulations of Eq. (8) show that a general initial profile splits into two waves moving in opposite directions with velocities $c = \pm 1$. However, under such straightforward averaging some interesting solutions are left out. In this paper we use more accurate averaging to obtain $\pi$-kink solutions moving with an arbitrary velocity.

By analogy with the Kapitza pendulum, see [17], one can expect that for small $\varepsilon$ both stationary solutions $\phi = 0$, $\phi = \pi$ are stable. Therefore, there may exist traveling wave solutions connecting these states. Below we will apply the normal form technique (see, e.g., [18,19]) to find such $\pi$-kink solutions.

2. Normal form calculations

The normal form technique is realized by applying a series of canonical near-identical transformations to move rapidly oscillating terms in the original equation (8) to higher order corrections. We will start with the Hamiltonian of Eq. (8) and reduce it to the Hamiltonian of double sine-Gordon equation up to higher-order terms. We will show that $\pi$-kink solutions obtained thereby are a good approximation of the solutions of the original equation.

The Hamiltonian of the parametrically forced SGE (8) is given by

$$H = \int_{-\infty}^{+\infty} \left( \frac{p^2}{2} + \frac{\phi_x^2}{2} - a \cos \phi \right) \, dx,$$  \hspace{1cm} (9)

where $p = \phi_t$ (below we omit the limits of integration and $dx$, wherever it is not confusing). One can easily obtain the equations of motion

$$\phi_t = \frac{\delta H}{\delta p} = p, \quad p_t = -\frac{\delta H}{\delta \phi} = \phi_{xx} - a \sin \phi,$$  \hspace{1cm} (10)

which are equivalent to the sine-Gordon equation. Let the first canonical transformation be defined implicitly by a generating function

$$S(p_1, \phi, t) = \int p_1 \phi \, dx + W_1(p_1, \phi, t)$$

as follows:

$$p = p_1 + \frac{\delta W_1}{\delta \phi}, \quad \phi_1 = \phi + \frac{\delta W_1}{\delta p_1}.$$

The new Hamiltonian is given by

$$H_1 = H + W_{1t} = \int \left[ \frac{1}{2} \left( p_1 + \frac{\delta W_1}{\delta \phi} \right)^2 + \frac{\phi_x^2}{2} - a \cos \phi \right] \, dx + W_{1t}.$$  \hspace{1cm} (12)
To kill the rapidly oscillating term in (12) we choose

\[ W_1 = \epsilon \int a_{-1} \cos \phi, \]

where \( a_{-1} \) is an antiderivative with zero average. The Hamiltonian takes the form

\[ H_1 = \int \frac{p_1^2}{2} + \frac{\phi_{1x}^2}{2} - \epsilon a_{-1} p_1 \sin \phi_1 + \epsilon^2 \frac{a_{-1}^2}{2} \sin^2 \phi_1. \]  

(13)

The last term in the above Hamiltonian cannot be removed by near-identity transformations since it has non-zero mean with respect to \( t \). However, all other terms with rapidly oscillating coefficients can be killed. Thus, choosing

\[ W_2 = \epsilon^2 \int a_{-2} p_2 \sin \phi_1 \]

we obtain the second transformation:

\[ p_1 = p_2 + \epsilon^2 a_{-2} p_2 \cos \phi_1, \quad \phi_2 = \phi_1 + \epsilon^2 a_{-2} \sin \phi_1. \]

The new Hamiltonian takes the form

\[ H_2 = H_1 + W_2, \]

\[ = \int \left( \frac{p_2^2}{2} + \frac{\phi_{2x}^2}{2} + \epsilon^2 \frac{a_{-2}^2}{2} \sin^2 \phi_1 \right) \sin \phi_1 + \epsilon^2 \frac{a_{-1}^2}{2} \sin^2 \phi_1 \]

\[ + \epsilon^2 \left( a_{-2}^2 p_2^2 \cos \phi_1 + \frac{(a_{-1}^2)}{2} \sin^2 \phi_1 - \epsilon a_{-1} a_{-2} p_2 \sin \phi_1 \cos \phi_1 + \epsilon^2 \frac{a_{-2}^2 p_2^2}{2} \cos^2 \phi_1 \right). \]

Solving the equation for \( \phi_1 \) we obtain

\[ \phi_1 = \phi_2 - \epsilon^2 a_{-2} \sin \phi_2 + \cdots \]

Substituting this expression in the Hamiltonian we express it in the new variables

\[ H_2 = \int \left( \frac{p_2^2}{2} + \frac{\phi_{2x}^2}{2} + \epsilon^2 \frac{a_{-2}^2}{2} \sin^2 \phi_2 \right) + \epsilon^2 \left( a_{-2}^2 \phi_2^2 \cos \phi_2 + a_{-2}^2 p_2^2 \cos \phi_2 + \frac{(a_{-1}^2)}{2} \sin^2 \phi_2 \right) + O(\epsilon^3). \]

The final transformation is generated by

\[ S_2 = \int p_3 \phi_2 + \epsilon^3 \left( a_{-3} \phi_{2x}^2 \cos \phi_2 - a_{-3} \phi_3^2 \cos \phi_2 - \frac{(a_{-1}^2)}{2} \sin^2 \phi_2 \right), \]

and is given implicitly by

\[ \phi_3 = \phi_2 - \epsilon^3 2a_{-3} p_3 \cos \phi_2, \]

\[ p_2 = p_3 + \epsilon^3 (-a_{-3} \sin \phi_2 \phi_{2x} - 2a_{-3} \cos \phi_2 \phi_{2xx} + a_{-3} \phi_3^2 \sin \phi_2 - \frac{1}{2} (a_{-1}^2) \sin 2 \phi_2). \]
We can again obtain an explicit expression for the transformation since the first equation contains no derivatives and can be inverted

\[ \phi_2 = \phi_3 + \epsilon^3 2a_{-3} p_3 \cos \phi_3 + \cdots \]

The transformed Hamiltonian takes the form

\[ H_3 = \int \left( \frac{p_3^2}{2} + \frac{3}{2} \phi_3^2 + \epsilon^2 \frac{(a_{-3})^2 \sin^2 \phi_3}{2} \right) + O(\epsilon^3), \]  

where \( \epsilon^3 \)-terms contain \( x \)-derivatives of \( \phi \) and \( p \) up to the second order. Therefore, the equations of motion take the form

\[ \phi_3t = p_3 + O(\epsilon^3), \quad p_3t = \phi_{3xx} - \frac{1}{2} \epsilon^2 (a_{-1}^2) \sin 2\phi_3 + O(\epsilon^3), \]  

where \( \epsilon^3 \)-terms contain \( x \)-derivatives of \( \phi \) and \( p \) up to the fourth order. After rescaling in Eq. (16)

\[ X = \epsilon x, \quad T = \epsilon t, \quad P = 2\epsilon^{-1} p_3, \quad \Phi = 2\phi_3, \]  

we obtain

\[ \Phi_T = P + O(\epsilon^2), \quad P_T = \Phi_{XX} - (a_{-1}^2) \sin \Phi + O(\epsilon). \]  

Evidently, the system (18) has \( 2\pi \)-kinks as approximate solutions. These solutions correspond to \( \pi \)-kinks in terms of \( (\phi_3, p_3) \), so that \( \phi_3 \approx U(x, t) \), where

\[ U(x, t) = 2 \arctan \left[ \exp \left( \epsilon \sqrt{(a_{-1}^2)} \frac{x - ct}{\sqrt{1 - c^2}} \right) \right]. \]  

3. Numerical simulations

We have performed numerical simulations of the original parametrically forced SGE (8) using the second-order leap-frog technique. We used the pure \( \pi \)-kink (19) to define the initial conditions for the variables \( \phi_3, p_3 \), as \( \phi_3(x, 0) = U(x, 0), p_3(x, 0) = U_t(x, 0) \). This gives us the initial conditions for the original variables \( \phi, p \) (in the leading order of \( \epsilon \)):

\[ \phi(x, 0) = U(x, 0), \quad p(x, 0) = U_t(x, 0) - \epsilon a_{-1}(0) \sin U(x, 0). \]  

In Fig. 1 we plot the numerical solution of Eq. (8) with the initial conditions (20) at the time \( t = 300 \) (dashed line) and the approximate analytical solution (19) (thin solid line). There is a good quantitative agreement between the theory and numerical simulations (the two curves in the figure almost coincide).

One can expect that after sufficiently long time the obtained approximate solution (19) will substantially deviate from the solution of the original equation (8). Then the perturbation analysis applied above fails to describe the solution, and taking into consideration higher orders in \( \epsilon \) is required. Indeed, as we can see from the numerical simulations, a “radiation” develops which leads, after a long but finite time, to complete destruction of the \( \pi \)-kink. In Fig. 2 one can already see a small radiation of two different harmonics, at \( t = 400 \). More detailed study of the mechanism of instability leading to the observed radiation will be done later.
Fig. 1. π-Kink solution in the parametrically forced SGE (8). The thick solid line corresponds to the initial profile given by (20) with \( c = 1/2, \alpha = \sin(t/\epsilon) \). The thin solid and dashed lines correspond to the approximate solution (19) and the result of the simulation, respectively, at \( t = 300 \). The analytical and numerical curves almost coincide. The parameters of the simulations are \( \epsilon = 0.1 \), time step \( dt = 0.01 \), mesh size \( dx = 0.05 \), and system size \( L = 500 \).

Fig. 2. Initial stage of the radiation of π-kink, at \( t = 400 \). Nearly flat part adjacent to the kink is enlarged to make the first harmonic of the radiation visible. The inset shows the second harmonic. The parameters of the simulations are the same as in Fig. 1.
4. Conclusions

In this paper we have found π-kink solutions to the periodically excited sine-Gordon equation with a fast mean-zero forcing. This was accomplished by reducing the original equation to the double sine-Gordon equation using the normal form technique.

We have obtained a good quantitative agreement between the theoretical results and numerical simulations of the original equation. We have observed in the simulations that after a long time the π-kink is destroyed by radiation due to a long term instability of the higher-order terms. The latter phenomenon will be investigated in the future by means of perturbation technique of integrable equations [10].

For quasi-one-dimensional ferromagnets the obtained π-kink solutions describe moving true domain walls generated by a rapidly oscillating external magnetic field. Returning to the original variables $t = i/C \sqrt{V H_0}$, $x = \bar{x}/\sqrt{V H_0}$, we can estimate the values of physical parameters which provide a realistic moving domain wall, using the experimental data from [12]. As a result, we have that for the field amplitude $H_0 \sim 1G$ and $\epsilon \sim 0.1$–0.01, the required field frequency and the wall width are $\omega \sim 10^9$ Hz and $\Delta x \sim 10 \mu$m, respectively.

References