Homework 7

In this homework, you will find a Lyapunov function and a decaying norm for linear differential equation with asymptotically stable equilibrium point. The last problem concerns construction of a norm in an asymptotically stable linear map in which the modified norm decays.

Problem # 1. Consider a matrix differential equation

\[
\dot{Z} = AZ + BZ, \quad Z(0) = C.
\]

Prove that it has a solution

\[
Z(t) = e^{At}Ce^{Bt}.
\]

Problem # 2. Consider the functional equation

\[
AX + XB = C.
\]

Prove that it has a solution

\[
X = -\int_0^\infty e^{At}Ce^{Bt}dt,
\]

provided the integral converges for any \( C \). Hint: Integrate the above differential equation on \((0, \infty)\) and show that \( Z(t) \to 0 \) as \( t \to \infty \).

Problem # 3. Now, consider the equation

\[
\dot{x} = Ax,
\]

where matrix \( A \) has eigenvalues with negative real parts. Construct Lyapunov function as follows: let

\[
V = x^T Y x
\]

with unknown symmetric matrix \( Y \). Differentiate and obtain

\[
\dot{V} = x^T (A^T Y + YA)x = -x^T x,
\]

by finding \( Y \) such that it solves the matrix equation

\[
A^T Y + YA = -I.
\]

You should obtain

\[
Y = \int_0^\infty e^{A^T t} e^{At} dt.
\]

Verify that the integral converges and that

\[
||x||_A = \sqrt{V(x)} = \sqrt{\int_0^\infty (e^{At}x)^T(e^{At}x)dt}
\]

is a norm which decreases along the solutions.
Problem # 4. For the mapping $x_{n+1} = Ax_n$, show that the modified norm

$$||x||_A = \sum_{n=0}^{\infty} ||A^n x||$$

has similar properties:

1. $||\cdot ||_A$ is a norm.
2. The induced operator norm of $A$ is less than 1: $||A||_A < 1$.

For the second item, you can use that all norms are equivalent in $\mathbb{R}^n$. 