Part IV: Duality is everywhere?

Knowns and unknowns in the world of topological modular forms

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What is duality?

Example

If V is a vector space over a field $\mathbb{F}$, its dual is $V^\vee = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$.

Definition (Algebra)

Let A be a ring. An A-module K is a dualizing A-module if

- For any A-module M
  - the double duality map $M \to \mathbf{Hom}_A(\mathbf{Hom}_A(M, K), K)$ is an equivalence
  - $\Leftrightarrow$ there is a perfect pairing $M \otimes_A \mathbf{Hom}_A(M, K) \to K$
- the double duality map $A \to \mathbf{Hom}_A(K, K)$ is an equivalence
- K has finite injective dimension.

K defines a duality functor $K : M \mapsto \mathbf{Hom}_A(M, K)$

Example

For $A = \mathbb{Z}$, $K = \mathbb{Z}$ is a dualizing module.

Duality in algebraic geometry

Let $(X, \mathcal{O}_X)$ be a scheme.

Definition (Algebraic geometry)

An $\mathcal{O}_X$-module $\mathcal{K}$ is a dualizing $\mathcal{O}_X$-module if

- the double duality map $\mathcal{O}_X \to \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{K})$ is an equivalence
- $\mathcal{K}$ has finite injective dimension.

Gives rise to Serre duality.

The projective line

Example

Let $\mathbb{P}^1 = \operatorname{Proj} A$, where $A = \mathbb{Z}[x_0, x_1]$ and $\deg x_i = 1$. Then the sheaf of Kähler differentials $\Omega^1_{\mathbb{P}^1}[1]$ is dualizing.

- $H^*(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \cong \mathbb{Z}$ in degree 1
- For any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$, there is a perfect pairing

$$H^s(\mathbb{P}^1, \mathcal{F}) \otimes H^{1-s}(\mathbb{P}^1, \mathbf{Hom}(\mathcal{F}, \Omega^1_{\mathbb{P}^1})) \to H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \cong \mathbb{Z}$$

- $\operatorname{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$, $m \in \mathbb{Z}$ corresponding to $\mathcal{O}(m) = \widetilde{(\Sigma^m A)}$
- $\Omega^1_{\mathbb{P}^1} \cong \mathcal{O}(-2)$

Duality in topology

Definition (Algebraic topology)

Let A be a ring spectrum. An A-module K is a dualizing A-module if

- the double duality map $A \to F_A(K, K)$ is an equivalence
- finiteness conditions:
  - For all i, $\pi_i K$ is a finitely generated $\pi_0 A$-module,
  - $\pi_i K = 0$ for $i \gg 0$, and
  - K has finite injective dimension: there is an integer n such that if M is an A-module with $\pi_i M = 0$ for $i > n$, then $\pi_i F_A(M, K) = 0$ for $i < 0$.

Example (a non-dualizing module)

The sphere S is not a dualizing S-module! It does not satisfy the finiteness conditions.

Brown-Comenetz duality

Definition

$$X \mapsto \mathrm{Hom}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$

is a cohomology theory, represented by a spectrum $I_{\mathbb{Q}/\mathbb{Z}}$. The resulting duality functor

$$X \mapsto I_{\mathbb{Q}/\mathbb{Z}}X = F(X, I_{\mathbb{Q}/\mathbb{Z}})$$

is called Brown-Comenetz duality.

Satisfies finiteness conditions, but not the double duality.

Example

Eilenberg-Maclane spectra are self-dual: $I_{\mathbb{Q}/\mathbb{Z}}H\mathbb{F}_p \simeq H\mathbb{F}_p$.

Anderson duality

Rational duality: $I_{\mathbb{Q}} \simeq H\mathbb{Q}$ represents $X \mapsto \mathrm{Hom}(\pi_{-*}X, \mathbb{Q})$.

Definition

The Anderson spectrum $I_{\mathbb{Z}}$ is the homotopy fiber of the natural map $I_{\mathbb{Q}} \to I_{\mathbb{Q}/\mathbb{Z}}$. The Anderson dual of X is $I_{\mathbb{Z}}X = F(X, I_{\mathbb{Z}})$.

For any X, fiber sequence $I_{\mathbb{Z}}X \to I_{\mathbb{Q}}X \to I_{\mathbb{Q}/\mathbb{Z}}X$ gives rise to a long exact sequence in homotopy. Interpret as a spectral sequence

$$\mathrm{Ext}^s_{\mathbb{Z}}(\pi_t X, \mathbb{Z}) \Rightarrow \pi_{-t-s} I_{\mathbb{Z}} X.$$

Example (Dualizing module)

$I_{\mathbb{Z}}$ is a dualizing S-module.

Self-dual objects

Example (Eilenberg-McLane spectra)

- $I_{\mathbb{Z}} H\mathbb{Z} \simeq H\mathbb{Z}$
- $I_{\mathbb{Z}} H\mathbb{F}_p \simeq \Sigma^{-1} H\mathbb{F}_p$

Example (Complex K-theory)

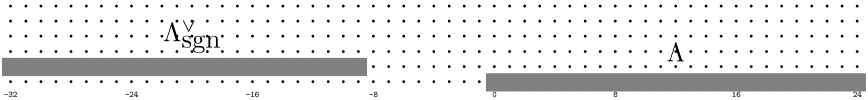
- $I_{\mathbb{Z}} K \simeq K$

We will see that KO, $Tmf(2)$, Tmf are also self-dual.

Question

How does this list continue?

Duality for $Tmf(2)$



$\pi_* Tmf(2) \cong \Lambda \oplus \Sigma^{-9}\Lambda^\vee_{\mathrm{sgn}}$

No torsion in π_* implies $\pi_* I_{\mathbb{Z}} Tmf(2) = \Lambda^\vee \oplus \Sigma^9 \Lambda_{\mathrm{sgn}}$

Theorem

The Anderson dual of $Tmf(2)$ is $\Sigma^9 Tmf(2)$.

How to descend the duality?

Recall

$Tmf(2)$ has an action by $G = GL_2(\mathbb{Z}/2)$ and

$$Tmf[1/2] \xrightarrow{\simeq} Tmf(2)^{hG}.$$

A toy example

K has an action of C_2 by complex conjugation, and $KO \simeq K^{hC_2}$.

Problem

Homotopy fixed points are a right adjoint; $I_{\mathbb{Z}} X = F(X^{hG}, I_{\mathbb{Z}})$ is generally inaccessible.

The norm map

In algebra

If M is a G-module, the norm is a map $\begin{array}{ccc} M/G & \longrightarrow & M^G \\ \uparrow & \nearrow_{x \mapsto \Sigma gx} & \\ M & & \end{array}$.

In homotopy theory

X is a spectrum with G-action, analogous norm map is

$$X_{hG} = (X \wedge_G EG_+) \xrightarrow{N} F_G(EG_+, X) = X^{hG}.$$

The Tate spectrum

The cofiber of the norm $X_{hG} \to X^{hG}$ is called the Tate spectrum and denoted X^{tG}.

If X is a ring spectrum, then so are X^{hG} and X^{tG}, and the map $X^{hG} \to X^{tG}$ is a ring map.

There is a spectral sequence

$$\hat{H}^s(G, \pi_t X) \Rightarrow \pi_{t-s} X^{tG}.$$

(All this and much more due to Greenlees-May.)

K-theory Tate spectrum

$\pi_* K = \mathbb{Z}[u^{\pm 1}]$, $C_2 = \{1, \sigma\}$ acts by ring maps and $\sigma u = -u$.

$$H^s(C_2, \pi_* K) = \mathbb{Z}[u^{\pm 2}, a, \eta]/(2a, 2\eta, \eta^2 - au^2) \Rightarrow \pi_* KO$$

Duality for KO

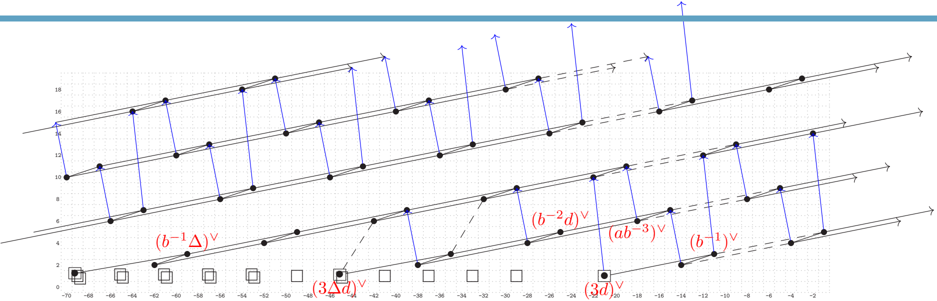
The norm map $K_{hC_2} \to K^{hC_2} \simeq KO$ is an equivalence $\rightsquigarrow$

$$I_{\mathbb{Z}}KO = F(KO, I_{\mathbb{Z}}) \simeq F(K_{hC_2}, I_{\mathbb{Z}}) \simeq F(K, I_{\mathbb{Z}})^{hC_2}$$

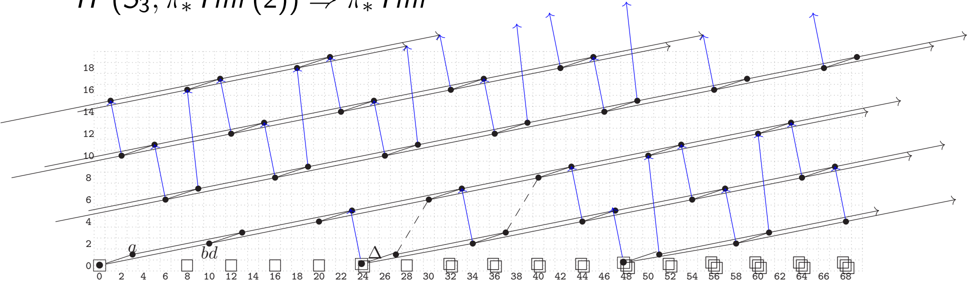
The C_2-action on $I_{\mathbb{Z}}K$ is twisted $\rightsquigarrow$ shifted differentials $\rightsquigarrow$

Proposition

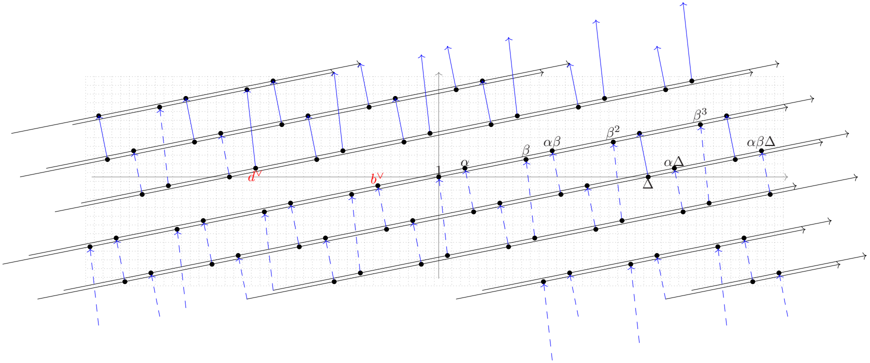
$$I_{\mathbb{Z}}KO \simeq \Sigma^4 KO$$



$H^*(S_3, \pi_* Tmf(2)) \Rightarrow \pi_* Tmf$



$$\hat{H}^*(S_3, \pi_* Tmf(2)) \Rightarrow \pi_* Tmf(2)^{tS_3}$$



Duality for Tmf

The norm map $Tmf(2)_{hS_3} \to Tmf(2)^{hS_3} \simeq Tmf$ is an equivalence $\rightsquigarrow$

$$I_{\mathbb{Z}} Tmf = F(Tmf, I_{\mathbb{Z}}) \simeq F(Tmf(2)_{hS_3}, I_{\mathbb{Z}}) \simeq F(Tmf(2), I_{\mathbb{Z}})^{hS_3}$$

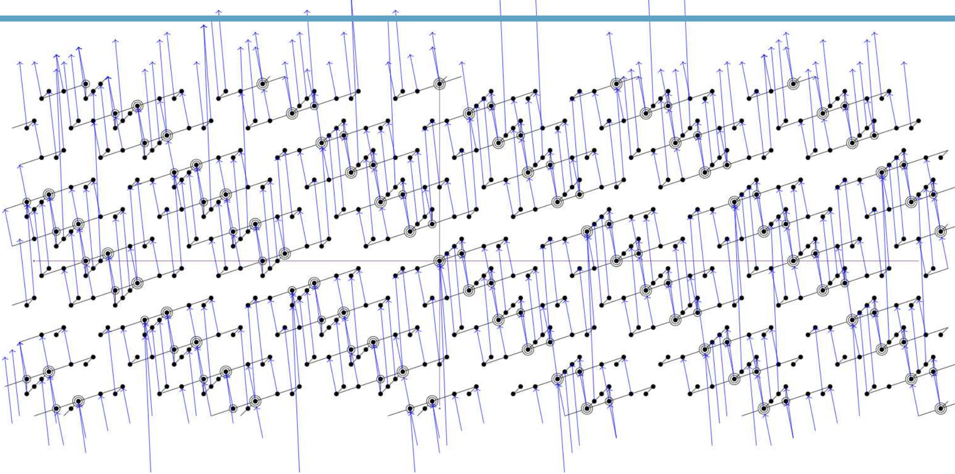
The S_3-action on $I_{\mathbb{Z}} Tmf(2)$ is twisted $\rightsquigarrow$ shifted differentials $\rightsquigarrow$

Theorem

$$I_{\mathbb{Z}} Tmf \simeq \Sigma^{21} Tmf$$

Unknowns

- Is $\Sigma^{21}\mathcal{O}$ a dualizing $\mathcal{O}$-module?
- What are other examples of this kind of duality?
  - Derived algebraic stacks with duality? Maybe some of the Shimura stacks of Behrens and Lawson? Would be very interesting at higher chromatic heights, but almost no known examples.
  - Other spectra with a finite group action and vanishing associated Tate spectrum?
  - What is the general principle?



Thank you!!!