Part IV: Duality is everywhere?
Knowns and unknowns in the world of topological modular forms

Vesna Stojanoska

Massachusetts Institute of Technology

Young Women in Topology
Bochum, July 6-8 2012
What is duality?

**Example**
If $V$ is a vector space over a field $\mathbb{F}$, its dual is $V^\vee = \text{Hom}_{\mathbb{F}}(V, \mathbb{F})$.

**Definition (Algebra)**
Let $A$ be a ring. An $A$-module $K$ is a dualizing $A$-module if
- For any $A$-module $M$
  - the double duality map $M \to \text{Hom}_A(\text{Hom}_A(M, K), K)$ is an equivalence
  - there is a perfect pairing $M \otimes_A \text{Hom}_A(M, K) \to K$
- the double duality map $A \to \text{Hom}_A(K, K)$ is an equivalence
- $K$ has finite injective dimension.

$K$ defines a duality functor $K : M \mapsto \text{Hom}_A(M, K)$

**Example**
For $A = \mathbb{Z}$, $K = \mathbb{Z}$ is a dualizing module.
Duality in algebraic geometry

Let \((X, \mathcal{O}_X)\) be a scheme.

**Definition (Algebraic geometry)**

An \(\mathcal{O}_X\)-module \(\mathcal{K}\) is a dualizing \(\mathcal{O}_X\)-module if

- the double duality map \(\mathcal{O}_X \to \text{Hom}_{\mathcal{O}_X}(\mathcal{K}, \mathcal{K})\) is an equivalence
- \(\mathcal{K}\) has finite injective dimension.

Gives rise to Serre duality.
Example

Let $\mathbb{P}^1 = \text{Proj} \ A$, where $A = \mathbb{Z}[x_0, x_1]$ and $\text{deg} \ x_i = 1$. Then the sheaf of Kähler differentials $\Omega^1_{\mathbb{P}^1}[1]$ is dualizing.

$\triangleright \ H^*(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \cong \mathbb{Z}$ in degree 1

$\triangleright \ $ For any quasi-coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1$, there is a perfect pairing

$$ H^s(\mathbb{P}^1, \mathcal{F}) \otimes H^{1-s}(\mathbb{P}^1, \text{Hom}(\mathcal{F}, \Omega^1_{\mathbb{P}^1})) \to H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \cong \mathbb{Z} $$

$\triangleright \ $ Pic$(\mathbb{P}^1) \cong \mathbb{Z}$, $m \in \mathbb{Z}$ corresponding to $\mathcal{O}(m) = (\Sigma^m A)$

$\triangleright \ $ $\Omega^1_{\mathbb{P}^1} \cong \mathcal{O}(-2)$
Definition (Algebraic topology)

Let $A$ be a ring spectrum. An $A$-module $K$ is a dualizing $A$-module if

- the double duality map $A \to F_A(K, K)$ is an equivalence
- finiteness conditions:
  - For all $i$, $\pi_i K$ is a finitely generated $\pi_0 A$-module,
  - $\pi_i K = 0$ for $i \gg 0$, and
  - $K$ has finite injective dimension: there is an integer $n$ such that if $M$ is an $A$-module with $\pi_i M = 0$ for $i > n$, then $\pi_i F_A(M, K) = 0$ for $i < 0$.

Example (a non-dualizing module)

The sphere $S$ is not a dualizing $S$-module! It does not satisfy the finiteness conditions.
Brown-Comenetz duality

Definition

\[ X \mapsto \text{Hom}(\pi_{\ast} X, \mathbb{Q}/\mathbb{Z}) \]

is a cohomology theory, represented by a spectrum \( I_{\mathbb{Q}/\mathbb{Z}} \). The resulting duality functor

\[ X \mapsto I_{\mathbb{Q}/\mathbb{Z}} X = F(X, I_{\mathbb{Q}/\mathbb{Z}}) \]

is called **Brown-Comenetz duality**.

Satisfies finiteness conditions, but not the double duality.

Example

Eilenberg-Maclane spectra are self-dual: \( I_{\mathbb{Q}/\mathbb{Z}} H\mathbb{F}_p \simeq H\mathbb{F}_p \).
Rational duality: $I_\mathbb{Q} \simeq H\mathbb{Q}$ represents $X \mapsto \text{Hom}(\pi_{-*}X, \mathbb{Q})$.

**Definition**

The Anderson spectrum $I_\mathbb{Z}$ is the homotopy fiber of the natural map $I_\mathbb{Q} \to I_\mathbb{Q}/\mathbb{Z}$. The Anderson dual of $X$ is $I_\mathbb{Z}X = F(X, I_\mathbb{Z})$.

For any $X$, fiber sequence $I_\mathbb{Z}X \to I_\mathbb{Q}X \to I_\mathbb{Q}/\mathbb{Z}X$ gives rise to a long exact sequence in homotopy. Interpret as a spectral sequence

$$\text{Ext}^s_\mathbb{Z}(\pi_tX, \mathbb{Z}) \Rightarrow \pi_{-t-s}I_\mathbb{Z}X.$$ 

**Example (Dualizing module)**

$I_\mathbb{Z}$ is a dualizing $S$-module.
Self-dual objects

Example (Eilenberg-McLane spectra)

- $l_{\mathbb{Z}} H \mathbb{Z} \simeq H \mathbb{Z}$
- $l_{\mathbb{Z}} H \mathbb{F}_p \simeq \Sigma^{-1} H \mathbb{F}_p$

Example (Complex $K$-theory)

- $l_{\mathbb{Z}} K \simeq K$

We will see that $KO$, $Tmf(2)$, $Tmf$ are also self-dual.

Question

*How does this list continue?*
Duality for $Tmf(2)$

\[ \pi_* Tmf(2) \cong \Lambda \oplus \Sigma^{-9} \Lambda_{sgn}^{\vee} \]

No torsion in $\pi_*$ implies $\pi_* I_\mathbb{Z} Tmf(2) = \Lambda^{\vee} \oplus \Sigma^9 \Lambda_{sgn}$

**Theorem**

*The Anderson dual of $Tmf(2)$ is $\Sigma^9 Tmf(2)$.*
How to descend the duality?

Recall

$Tmf(2)$ has an action by $G = GL_2(\mathbb{Z}/2)$ and

$$Tmf[1/2] \xrightarrow{\sim} Tmf(2)^{hG}.$$ 

A toy example

$K$ has an action of $C_2$ by complex conjugation, and $KO \simeq K^{hC_2}$.

Problem

Homotopy fixed points are a right adjoint; $I_{\mathbb{Z}}X = F(X^{hG}, I_{\mathbb{Z}})$ is generally inaccessible.
The norm map

In algebra
If $M$ is a $G$-module, the norm is a map $M/G \rightarrow M^G$.

In homotopy theory
$X$ is a spectrum with $G$-action, analogous norm map is

$$X_{hG} = (X \wedge_G EG_+) \overset{N}{\rightarrow} F_G(EG_+, X) = X^{hG}.$$
The Tate spectrum

The cofiber of the norm $X_{hG} \to X^{hG}$ is called the Tate spectrum and denoted $X^{tG}$.

If $X$ is a ring spectrum, then so are $X^{hG}$ and $X^{tG}$, and the map $X^{hG} \to X^{tG}$ is a ring map.

There is a spectral sequence

$$\hat{H}^s(G, \pi_t X) \Rightarrow \pi_{t-s} X^{tG}.$$ 

(All this and much more due to Greenlees-May.)
$K$-theory Tate spectrum

$$\pi_* K = \mathbb{Z}[u^{\pm 1}], \ C_2 = \{1, \sigma\} \text{ acts by ring maps and } \sigma u = -u.$$ 

$$H^s(C_2, \pi_* K) = \mathbb{Z}[u^{\pm 2}, a, \eta]/(2a, 2\eta, \eta^2 - au^2) \Rightarrow \pi_* KO$$
The norm map $K_{hC_2} \rightarrow K^{hC_2} \cong KO$ is an equivalence

$$I_{\mathbb{Z}}KO = F(KO, I_{\mathbb{Z}}) \cong F(K_{hC_2}, I_{\mathbb{Z}}) \cong F(K, I_{\mathbb{Z}})^{hC_2}$$

The $C_2$-action on $I_{\mathbb{Z}}K$ is twisted $\leadsto$ shifted differentials $\leadsto$

Proposition

$$I_{\mathbb{Z}}KO \cong \Sigma^4 KO$$
$H^*(S_3, \pi_* Tmf(2)) \Rightarrow \pi_* Tmf$
\[ \hat{H}^*(S_3, \pi_* Tmf(2)) \Rightarrow \pi_* Tmf(2)^{tS_3} \]
Duality for Tmf

The norm map \( Tmf(2)_{hS_3} \rightarrow Tmf(2)^{hS_3} \cong Tmf \) is an equivalence

\[ l_{\mathbb{Z}}Tmf = F(Tmf, l_{\mathbb{Z}}) \cong F(Tmf(2)_{hS_3}, l_{\mathbb{Z}}) \cong F(Tmf(2), l_{\mathbb{Z}})^{hS_3} \]

The \( S_3 \)-action on \( l_{\mathbb{Z}}Tmf(2) \) is twisted \( \rightsquigarrow \) shifted differentials

Theorem

\[ l_{\mathbb{Z}}Tmf \cong \Sigma^{21} Tmf \]
Unknons

- Is $\Sigma^{21}\mathcal{O}$ a dualizing $\mathcal{O}$-module?
- What are other examples of this kind of duality?
  - Derived algebraic stacks with duality? Maybe some of the Shimura stacks of Behrens and Lawson? Would be very interesting at higher chromatic heights, but almost no known examples.
  - Other spectra with a finite group action and vanishing associated Tate spectrum?
- What is the general principle?
Thank you!!!