Part III: Shapes and forms of topological modular forms

Knowns and unknowns in the world of topological modular forms

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From algebraic geometry to homotopy theory

Rings in families $\rightsquigarrow$ algebraic geometry.
Use ring spectra instead!

Working definition (Lurie's theorem)

Let $\mathcal{X}$ be a stack, $\mathcal{O}$ a sheaf of ring spectra on its étale site. Then $(\mathcal{X}, \mathcal{O})$ is a derived, or spectral, Deligne-Mumford stack if

- $(\mathcal{X}, \pi_0\mathcal{O})$ is an ordinary Deligne-Mumford stack
- $\pi_i\mathcal{O}$ is a quasi-coherent sheaf on $(\mathcal{X}, \pi_0\mathcal{O})$.

Global sections

For a sheaf of ring spectra $\mathcal{F}$ on $\mathcal{X}$, define the homotopy global sections

$$R\Gamma\mathcal{F} = \mathcal{F}(\mathcal{X}) = \mathrm{holim}_{\mathcal{U}}\,\mathcal{F},$$

where $\mathcal{U}$ is the category of opens in $\mathcal{X}$.

$\mathcal{F}(\mathcal{X})$ is a ring spectrum, and there is a descent spectral sequence

$$H^s(\mathcal{X}, \pi_t\mathcal{F}) \Rightarrow \pi_{t-s}\mathcal{F}(\mathcal{X}).$$

Realization problem

Question

Given an ordinary stack $(\mathcal{X}, \mathcal{O}_0)$, can we derive it?

There is a general theorem by Lurie solving the realization problem for nice stacks.

Derived moduli of elliptic curves

Theorem (Goerss-Hopkins-Miller, Lurie)

The moduli stack of generalized elliptic curves $\bar{\mathcal{M}}$ is realizable, i.e. it admits a sheaf of ring spectra $\mathcal{O}$ such that for an étale map $Spec\,R \to \bar{\mathcal{M}}$ classifying a generalized elliptic curve C/R, the sections $E = \mathcal{O}(Spec\,R)$ form a complex oriented ring spectrum E with

- *$\pi_0 E = R$, and*
- *the formal group associated to E isomorphic to the completion of C at the identity.*

Moreover, $\pi_{2t}\mathcal{O} \cong \omega^t$, and $\pi_{2t+1}\mathcal{O} = 0$ as $\pi_0\mathcal{O}$-modules.

Remark

Lurie's general theorem only applies to $\mathcal{M}$; special care is needed for the compactification.

Shapes of topological modular forms

- $TMF := \mathcal{O}(\mathcal{M})$

  Periodic version
- $Tmf := \mathcal{O}(\bar{\mathcal{M}})$

  Self-dual version
- tmf is a connective cover of either TMF or Tmf

  Morally, tmf is global sections over $\mathcal{M}_{weier}$

Can use the descent spectral sequence to compute their homotopy groups.

Level structures

The forgetful map $f : \mathcal{M}(n) \to \mathcal{M}[1/n]$ is étale $\rightsquigarrow$ restrict $\mathcal{O}$ to $\mathcal{M}(n)$, and take its homotopy global sections to obtain $TMF(n)$.

Recall

The maps $f : \bar{\mathcal{M}}(n) \to \bar{\mathcal{M}}[1/2n]$ are not étale, but tamely ramified at the cusps.

Theorem

The compactified stacks $\bar{\mathcal{M}}(n)$ over $\mathbb{Z}[1/2n]$ admit étale sheaves of ring spectra $\mathcal{O}(n)$ with properties like those for $(\bar{\mathcal{M}}, \mathcal{O})$.
Define $Tmf(n) := \mathcal{O}(n)(\bar{\mathcal{M}}(n))$.

Nice connective versions automatically exist only if the genus of $\bar{\mathcal{M}}(n)$ is zero, which is only true for $n \leq 5$.

Galois descent

Recall

The group $GL_2(\mathbb{Z}/n)$ acts nicely on $\bar{\mathcal{M}}(n)$.

Theorem

There is a natural action of $GL_2(\mathbb{Z}/n)$ on $Tmf(n)[1/2]$, such that the map on global sections

$$Tmf[1/2n] \to Tmf(n)[1/2]$$

is the inclusion of homotopy fixed points.

Corollary

There is a homotopy fixed point spectral sequence

$$H^s\left(GL_2(\mathbb{Z}/n), \pi_t Tmf(n)[1/2]\right) \Rightarrow \pi_{t-s} Tmf[1/2n].$$

Let's compute some level 2 structures

$\mathcal{M}_{weier}[1/2]$ is represented by $\left(A_b = \mathbb{Z}[1/2][b_2, b_4, b_6], A_b[u^{\pm 1}, r]\right)$ with

$$C: \qquad y^2 = f_b(x) = x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4}.$$

$\bar{\mathcal{M}}[1/2]$ is the locus where the ideal (c_4, Δ) contains a unit.

- For a point P on C, $[2]P = 0 \Leftrightarrow P = [-1]P$
- $[-1](x, y) = (x, -y)$
- Specify a level 2 structure on $C \equiv$ specify roots x_0, x_1, x_2 of f_b
- Move x_0 to 0 $\quad\rightsquigarrow\quad C_\lambda : y^2 = x(x - \lambda_0)(x - \lambda_1)$
- Unique up to $x \mapsto u^{-2}x$, $y \mapsto u^{-3}y$ (and $\lambda_i \mapsto u^2\lambda$)
- C_λ is smooth if $0, \lambda_0, \lambda_1$ are all distinct
- C_λ has at most nodal singularity if not both λ_i are zero

Explicit level 2 structures

Proposition

Let Λ be the graded ring $\mathbb{Z}[1/2][\lambda_0, \lambda_1]$, where λ_i are in degree 2. Then

$$\bar{\mathcal{M}}(2) \simeq Proj\,\Lambda.$$

The invariant differential $\eta_C = \frac{dx}{2y}$ is in degree 1 ($\eta_C \mapsto u\eta_C$).
Implies that $\omega_{\bar{\mathcal{M}}(2)} \cong \mathcal{O}(1)$.

Still computing

$$\begin{array}{ccc} U = \operatorname{Spec}\Lambda - 0 & \hookrightarrow & \operatorname{Spec}\Lambda \\ \big\downarrow {\scriptstyle //\mathbb{G}_m} & & \big\downarrow {\scriptstyle //\mathbb{G}_m} \\ \operatorname{Proj}\Lambda & \longrightarrow & \operatorname{Spec}\Lambda//\mathbb{G}_m \end{array}$$

Local cohomology sequence

The point 0 is defined by the vanishing of the ideal (λ_0, λ_1), and there is a long exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0_{(\lambda_i)}(\operatorname{Spec}\Lambda) \longrightarrow H^0(\operatorname{Spec}\Lambda) \longrightarrow H^0(U) \longrightarrow \\ \longrightarrow H^1_{(\lambda_i)}(\operatorname{Spec}\Lambda) \longrightarrow H^1(\operatorname{Spec}\Lambda) \longrightarrow H^1(U) \longrightarrow \\ \longrightarrow H^2_{(\lambda_i)}(\operatorname{Spec}\Lambda) \longrightarrow 0. \end{aligned}$$

Still computing

λ_0, λ_1 is a regular sequence in $\Lambda \rightsquigarrow$ the local cohomology groups $H^*_{(\lambda_i)}(\operatorname{Spec}\Lambda)$ are computed as the cohomology of the Koszul complex

$$\Lambda \to \Lambda[1/\lambda_0] \times \Lambda[1/\lambda_1] \to \Lambda[1/\lambda_0\lambda_1]$$

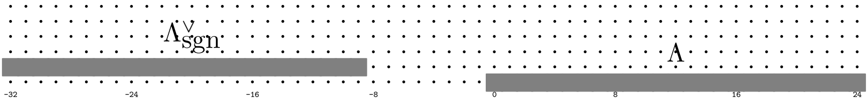
which is $\Lambda/(\lambda_0^\infty, \lambda_1^\infty)$ concentrated in degree 2.

Cohomology

$$H^s(\bar{\mathcal{M}}(2), \omega^*) = \begin{cases} \Lambda, & \text{for } s = 0 \\ \Lambda/(\lambda_0^\infty, \lambda_1^\infty), & \text{for } s = 1 \\ 0, & \text{else.} \end{cases}$$

Descent for $Tmf(2)$

$$H^s(\bar{\mathcal{M}}(2), \omega^t) \Rightarrow \pi_{2t-s} Tmf(2).$$



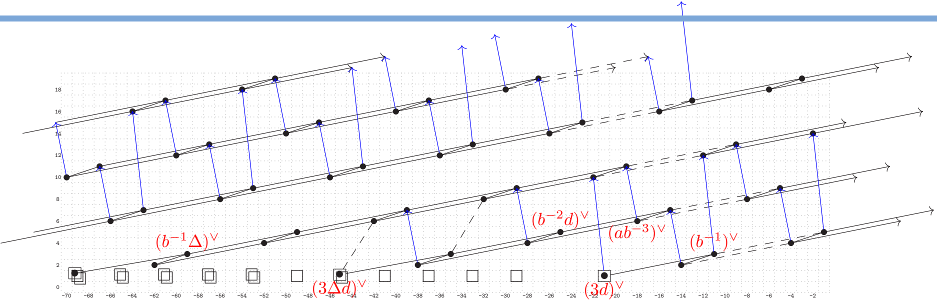
Finally

$$\pi_* Tmf(2) = \Lambda \oplus \Sigma^{-9}\Lambda/(\lambda_0^\infty, \lambda_1^\infty).$$

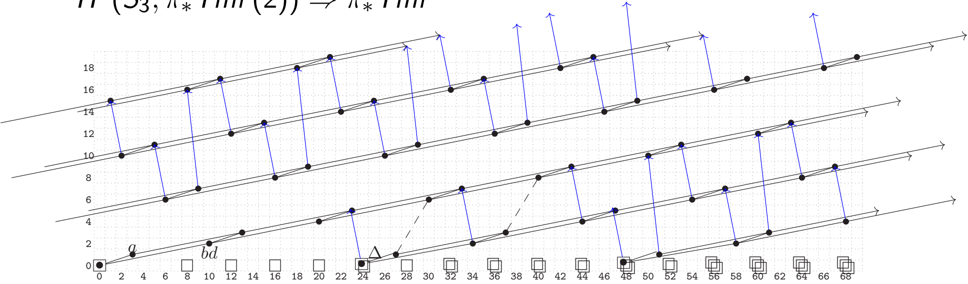
Galois descent

The group $GL_2(\mathbb{Z}/2)$ is the symmetric group S_3, acting by permuting the three non-trivial points of order two on C.

Can compute explicitly the action of S_3 on $\pi_* Tmf(2)$, and the resulting group cohomology.



$H^*(S_3, \pi_* Tmf(2)) \Rightarrow \pi_* Tmf$



Unknowns

Open questions

- Is $\mathcal{O}$ really a sheaf for the log-étale topology on $\bar{\mathcal{M}}$?
  - Hill-Lawson
- If X is a space, or maybe manifold, describe $tmf^*(X)$ geometrically.
  - $tmf^*(X)$ should consist of *2-dimensional conformal field theories over X*
  - Bartels-Douglas-Henriques, Stolz-Teichner
- Understand the structure of $tmf \wedge tmf$
  - Behrens-Ormsby-Stapleton-S.
- Understand the category of tmf-modules
  - Meier
- ... and a lot more...