Part II: Elliptic curves for the homotopy theorist

Knowns and unknowns in the world of topological modular forms

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Recall

- $\pi^S_* \to \Omega^G_* \to E^{-*}$
- $\Omega^U_*$ has a universal formal group law (of dimension one)
- Height filtration
- Need formal group laws of various heights
  - Simplest method: Complete a one-dimensional algebraic group at its identity section
  - Only examples: $\mathbb{G}_a$, $\mathbb{G}_m$, and elliptic curves
What is an elliptic curve?

**Definition**
An elliptic curve over a base scheme $S$ is a pair of morphisms $p : C \leftrightarrow S : e$, such that

- $p$ is flat, proper, smooth morphism of relative dimension one, such that each fiber is a curve of genus one, and
- $e$ is a section of $p$.

**Theorem**
An elliptic curve $C/S$ has a unique group structure with $e$ as the identity.
The group structure

Let $p : C \equiv S : e$ be an elliptic curve, $T$ an $S$-scheme. If $P, Q, R$ are $T$-points of $C$, then

$$P + Q = R$$

iff there is an invertible sheaf $\mathcal{L}_0$ on $T$ and an iso

$$\mathcal{I}^{-1}(P) \otimes \mathcal{I}^{-1}(Q) \otimes \mathcal{I}(e) \cong \mathcal{I}^{-1}(R) \otimes p^*_T(\mathcal{L}_0).$$

In other words,

$$P \mapsto \mathcal{I}^{-1}(P) \otimes \mathcal{I}(e)$$

$$C(T) \xleftrightarrow{1-1} \text{Pic}^{(0)}(C_T/T) = \begin{cases} \text{the abelian group of iso classes} \\ \text{of invertible sheaves on } C_T, \text{ which are} \\ \text{fiberwise of degree zero, modulo those} \\ \text{of form } p^* T(\mathcal{L}_0) \end{cases}$$
Invariant differentials

Let $p : C \leftrightarrow S : e$ be an elliptic curve, and $\Omega^1_{C/S}$ the sheaf of Kähler differentials on $C$.

**Definition**
The sheaf of invariant differentials on $S$ is

$$\omega_C = p_*\Omega^1_{C/S} = p_*\mathcal{I}(e)/\mathcal{I}(e)^2.$$

$\omega_C$ is an invertible line bundle, and locally on $S$ we can choose a generator $\eta$. 
Riemann-Roch theorem $\Rightarrow$ For $n \geq 1$, $p_*\mathcal{I}^{-n}(e)$ is locally free of rank $n$.

Having chosen a generator of $\omega_C$, we have:

- $p_*\mathcal{I}^{-2}(e)$ free on 1, $x$
  - $x$ is unique up to $x \mapsto u^{-2}x + r$
- $p_*\mathcal{I}^{-3}(e)$ free on 1, $x, y$
  - $y$ is unique up to $y \mapsto u^{-3}y + u^{-2}sx + t$
- $p_*\mathcal{I}^{-4}(e)$ free on 1, $x, y, x^2$
- $p_*\mathcal{I}^{-5}(e)$ free on 1, $x, y, x^2, xy$
- $p_*\mathcal{I}^{-6}(e)$ free on 1, $x, y, x^2, xy, x^3$ or 1, $x, y, x^2, xy, y^2$
Weierstrass equations

Hence a relation

\[ C_{\text{weier}} : \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \]

Conversely, any smooth Weierstrass curve is elliptic.

Example

- Legendre curves: \( y^2 = x(x - 1)(x - \lambda) \)
- Tate normal curves: \( y^2 + \alpha xy + y = x^3 \)
- Nodal curve: \( y^2 = x^3 + x \)
- Cusp curve: \( y^2 = x^3 \)
Weierstrass moduli stack

Classify

\[ C_{\text{weier}} : \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \]

modulo change of variables

\[ \eta_R : x \mapsto u^{-2} x + r \]
\[ y \mapsto u^{-3} y + u^{-2} s x + t \]

\[ A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] , \quad \Gamma = A[u^{\pm 1}, r, s, t] \]

\((A, \Gamma)\) is a Hopf algebroid: \( A \xrightarrow{\eta_L} \Gamma \xleftarrow{\eta_R} \Gamma \)

The moduli stack of Weierstrass curves \( \mathcal{M}_{\text{weier}} \) is the stackification (homotopy coequalizer) of \((A, \Gamma)\)

\[(\text{Spec } \Gamma \rightrightarrows \text{Spec } A) \to \mathcal{M}_{\text{weier}}\]
Some invariants

\[(A, \Gamma)[1/2] \simeq (A_b = \mathbb{Z}[1/2][b_2, b_4, b_6], A_b[u^\pm 1, r]) \text{ with} \]
\[C_{\text{weier}} : \quad y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6\]

\[(A, \Gamma)[1/6] \simeq (A_c = \mathbb{Z}[1/6][c_4, c_6], A_c[u^\pm 1]) \simeq \text{Proj } \mathbb{Z}[1/6][c_4, c_6] \text{ with} \]
\[C_{\text{weier}} : \quad y^2 = x^3 - 27c_4x - 54c_6\]

\[\Delta = \frac{c_4^3 - c_6^2}{12^3} \quad j = \frac{c_4^3}{\Delta}\]

c_i, \Delta, j are defined for any Weierstrass curve.
Under a generic linear transformation,
\[c_i \mapsto u^i c_i \quad \Delta \mapsto u^{12}\Delta \quad j \mapsto j\]
Moduli stacks of elliptic curves

\( C_{weier} \) is smooth if and only if \( \Delta \) is invertible, implying

Proposition

The moduli stack of elliptic curves \( \mathcal{M} \) is

\[
\mathcal{M} = \text{Stack}(A[\Delta^{-1}], \Gamma[\Delta^{-1}]).
\]

Consider \( j : \mathcal{M} \to \mathbb{A}^1 \). To compactify, allow \( j = \infty \).

The moduli stack of generalized elliptic curves \( \tilde{\mathcal{M}} \) is the substack of \( \mathcal{M}_{weier} \) defined by the invertibility of the ideal \((c_4, \Delta)\).

\( \tilde{\mathcal{M}} \) classifies curves of genus one which are smooth or have a nodal singularity.
Modular forms

\[ \omega : (S \xrightarrow{C} \overline{M}) \mapsto \omega_C \] is an invertible line bundle on \( \mathcal{M} \) as well as \( \overline{M} \).

**Definition**

The ring of modular forms \( MF_* \) is the graded ring of global sections

\[ H^0(\overline{M}, \omega^*) \].

**Exercise**

\[ MF_* = \mathbb{Z}[c_4, c_6, \Delta] / (12^3 \Delta = c_4^3 - c_6^2). \]
Level structures

If $C/S$ is a smooth elliptic curve over a base on which $n$ is invertible,

$$C[n] \cong (\mathbb{Z}/n)^2.$$ 

**Definition**

- A $\Gamma(n)$ or level $n$ structure is an iso $(\mathbb{Z}/n)^2 \to C[n]$ 
- A $\Gamma_1(n)$-structure is an injection $\mathbb{Z}/n \hookrightarrow C[n]$ 
- A $\Gamma_0(n)$-structure is a cyclic subgroup $H \hookrightarrow C$ of order $n$

The corresponding moduli stacks are $\mathcal{M}(n)$, $\mathcal{M}_1(n)$, $\mathcal{M}_0(n)$. 
Level $n$ structures at the cusps

Let $C_0$ be the nodal generalized elliptic curve. Then

$$C_0 \cong \mathbb{P}^1/(0 \sim \infty) \quad \Rightarrow \quad C_0[n] \cong \mu_n.$$ 

Replace $C$ by $\tilde{C} \cong ((\mathbb{Z}/n) \times \mathbb{P}^1) / (\sim)$. Then $\tilde{C}[n] \cong (\mathbb{Z}/n)^2$.
Group action

- $GL_2(\mathbb{Z}/n) = \text{Aut}(\mathbb{Z}/n)^2$ acts freely and transitively on level $n$ structures of a smooth elliptic curve $C$

- For singular curves, $U = \left\{ \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}$ acts trivially

Forget the level structure $\leadsto$

$$f : \tilde{\mathcal{M}}(n) \to \tilde{\mathcal{M}}[1/n].$$

Over the smooth locus, $f$ is a $GL_2(\mathbb{Z}/n)$-torsor. Over the cusp, ramified of degree $2n$. 