Part II: Elliptic curves for the homotopy theorist

Knowns and unknowns in the world of topological modular forms

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Young Women in Topology
Bochum, July 6-8 2012

Recall

- $\pi_*^s \to \Omega_*^G \to E^{-*}$
- Ω_*^U has a universal formal group law (of dimension one)
- Height filtration
- Need formal group laws of various heights
  - Simplest method: Complete a one-dimensional algebraic group at its identity section
  - Only examples: $\mathbb{G}_a$, $\mathbb{G}_m$, and elliptic curves

What is an elliptic curve?

Definition

An elliptic curve over a base scheme S is a pair of morphisms $p : C \leftrightarrows S : e$, such that

- p is flat, proper, smooth morphism of relative dimension one, such that each fiber is a curve of genus one, and
- e is a section of p.

Theorem

An elliptic curve C/S has a unique group structure with e as the identity.

The group structure

Let $p : C \leftrightarrows S : e$ be an elliptic curve, T an S-scheme.
If P, Q, R are T-points of C, then

$$P + Q = R$$

iff there is an invertible sheaf $\mathcal{L}_0$ on T and an iso

$$\mathcal{I}^{-1}(P) \otimes \mathcal{I}^{-1}(Q) \otimes \mathcal{I}(e) \cong \mathcal{I}^{-1}(R) \otimes p_T^*(\mathcal{L}_0).$$

In other words,

$$P \mapsto \mathcal{I}^{-1}(P) \otimes \mathcal{I}(e)$$

$$C(T) \overset{1-1}{\longleftrightarrow} \mathrm{Pic}^{(0)}(C_T/T) = \begin{cases} \text{the abelian group of iso classes} \\ \text{of invertible sheaves on } C_T\text{, which are} \\ \text{fiberwise of degree zero, modulo those} \\ \text{of form } p^*T(\mathcal{L}_0) \end{cases}$$

Invariant differentials

Let $p : C \leftrightarrows S : e$ be an elliptic curve, and $\Omega^1_{C/S}$ the sheaf of Kähler differentials on C.

Definition

The sheaf of invariant differentials on S is

$$\omega_C = p_* \Omega^1_{C/S} = p_* \mathcal{I}(e)/\mathcal{I}(e)^2.$$

ω_C is an invertible line bundle, and locally on S we can choose a generator η.

Weierstrass equations

Riemann-Roch theorem $\Rightarrow$ For $n \geq 1$, $p_* \mathcal{I}^{-n}(e)$ is locally free of rank n.

Having chosen a generator of ω_C, we have:

- $p_* \mathcal{I}^{-2}(e)$ free on $1, x$
  - x is unique up to $x \mapsto u^{-2}x + r$
- $p_* \mathcal{I}^{-3}(e)$ free on $1, x, y$
  - y is unique up to $y \mapsto u^{-3}y + u^{-2}sx + t$
- $p_* \mathcal{I}^{-4}(e)$ free on $1, x, y, x^2$
- $p_* \mathcal{I}^{-5}(e)$ free on $1, x, y, x^2, xy$
- $p_* \mathcal{I}^{-6}(e)$ free on $1, x, y, x^2, xy, x^3$ or $1, x, y, x^2, xy, y^2$

Weierstrass equations

Hence a relation

$$C_{weier}: \qquad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Conversely, any smooth Weierstrass curve is elliptic.

Example

- Legendre curves: $y^2 = x(x-1)(x-\lambda)$
- Tate normal curves: $y^2 + \alpha xy + y = x^3$
- Nodal curve: $y^2 = x^3 + x$
- Cusp curve: $y^2 = x^3$

Weierstrass moduli stack

Classify

$$C_{weier}: \qquad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

modulo change of variables

$$\begin{aligned} \eta_R : x &\mapsto u^{-2}x + r \\ y &\mapsto u^{-3}y + u^{-2}sx + t \end{aligned}$$

$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$, $\Gamma = A[u^{\pm 1}, r, s, t]$

(A, Γ) is a Hopf algebroid: $A \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} \Gamma$

The moduli stack of Weierstrass curves $\mathcal{M}_{weier}$ is the stackification (homotopy coequalizer) of (A, Γ)

$$(\operatorname{Spec}\Gamma \rightrightarrows \operatorname{Spec}A) \to \mathcal{M}_{weier}$$

Some invariants

$(A, \Gamma)[1/2] \simeq \left(A_b = \mathbb{Z}[1/2][b_2, b_4, b_6], A_b[u^{\pm 1}, r]\right)$ with

$$C_{weier}: \qquad y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

$(A, \Gamma)[1/6] \simeq \left(A_c = \mathbb{Z}[1/6][c_4, c_6], A_c[u^{\pm 1}]\right) \simeq \operatorname{Proj} \mathbb{Z}[1/6][c_4, c_6]$ with

$$C_{weier}: \qquad y^2 = x^3 - 27c_4x - 54c_6$$

$$\Delta = \frac{c_4^3 - c_6^2}{12^3} \qquad\qquad j = \frac{c_4^3}{\Delta}$$

c_i, Δ, j are defined for *any* Weierstrass curve.
Under a generic linear transformation,

$$c_i \mapsto u^i c_i \qquad\qquad \Delta \mapsto u^{12}\Delta$$

$$j \mapsto j$$

Moduli stacks of elliptic curves

C_{weier} is smooth if and only if Δ is invertible, implying

Proposition

The moduli stack of elliptic curves $\mathcal{M}$ is

$$\mathcal{M} = Stack\big(A[\Delta^{-1}], \Gamma[\Delta^{-1}]\big).$$

Consider $j : \mathcal{M} \to \mathbb{A}^1$. To compactify, allow $j = \infty$.

The moduli stack of generalized elliptic curves $\bar{\mathcal{M}}$ is the substack of $\mathcal{M}_{weier}$ defined by the invertibility of the ideal (c_4, Δ).

$\bar{\mathcal{M}}$ classifies curves of genus one which are smooth or have a nodal singularity.

Modular forms

$\omega : (S \xrightarrow{C} \bar{\mathcal{M}}) \mapsto \omega_C$ is an invertible line bundle on $\mathcal{M}$ as well as $\bar{\mathcal{M}}$.

Definition

The ring of modular forms MF_* is the graded ring of global sections

$$H^0(\bar{\mathcal{M}}, \omega^{\otimes *}).$$

Exercise

$MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(12^3\Delta = c_4^3 - c_6^2)$.

Level structures

If C/S is a smooth elliptic curve over a base on which n is invertible,

$$C[n] \cong (\mathbb{Z}/n)^2.$$

Definition

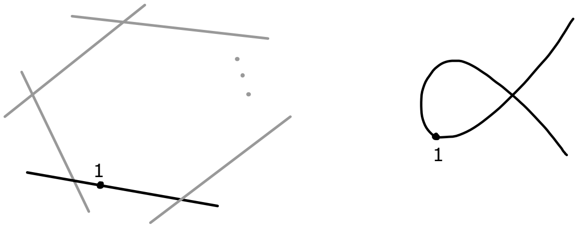
- A $\Gamma(n)$ or level n structure is an iso $(\mathbb{Z}/n)^2 \to C[n]$
- A $\Gamma_1(n)$-structure is an injection $\mathbb{Z}/n \hookrightarrow C[n]$
- A $\Gamma_0(n)$-structure is a cyclic subgroup $H \hookrightarrow C$ of order n

The corresponding moduli stacks are $\mathcal{M}(n)$, $\mathcal{M}_1(n)$, $\mathcal{M}_0(n)$.

Level n structures at the cusps

Let C_0 be the nodal generalized elliptic curve. Then

$$C_0 \cong \mathbb{P}^1/(0 \sim \infty) \qquad \Rightarrow \qquad C_0[n] \cong \mu_n.$$



Replace C by $\tilde{C} \cong \left((\mathbb{Z}/n) \times \mathbb{P}^1\right)/(\sim)$. Then $\tilde{C}[n] \cong (\mathbb{Z}/n)^2$.
$\rightsquigarrow \quad \bar{\mathcal{M}}(n)$

Group action

- $GL_2(\mathbb{Z}/n) = \mathrm{Aut}(\mathbb{Z}/n)^2$ acts freely and transitively on level n structures of a smooth elliptic curve C
- For singular curves, $U = \left\{ \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}$ acts trivially

Forget the level structure $\rightsquigarrow$

$$f : \bar{\mathcal{M}}(n) \to \bar{\mathcal{M}}[1/n].$$

Over the smooth locus, f is a $GL_2(\mathbb{Z}/n)$-torsor. Over the cusp, ramified of degree $2n$.