Part I: Intro
Genera and the stable homotopy groups of spheres

Knowns and unknowns in the world of topological modular forms

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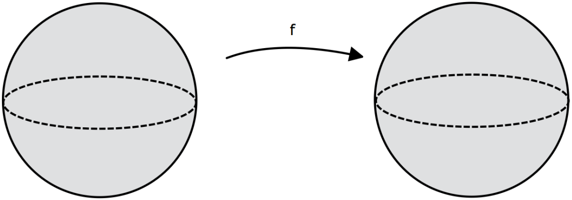
The motivating unknown: homotopy groups of spheres

Question

What are all maps $f : S^{n+k} \to S^n$, up to homotopy? Stably?

Example

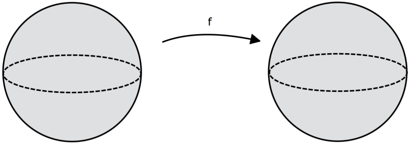
If $k = 0$, f is determined by its degree.



If x is a regular value of f, the degree of f equals the cardinality of $f^{-1}(x)$.

Manifolds in spheres

What if $k > 0$?



$f^{-1}(x)$ is a k-manifold embedded in S^{n+k}, equipped with a framing of its normal bundle.

$f_0 \simeq f_1 \quad \rightsquigarrow \quad$ a (framed) cobordism between $f_0^{-1}(x)$ and $f_1^{-1}(x)$.

Theorem (Pontryagin)

The homotopy class of f is completely determined by the geometry of the inverse image f^{-1} of a small neighborhood of a regular value. Namely,

$$\pi_{n+k}(S^n) \simeq \Omega_k^{fr}(S^{n+k}).$$

And more manifolds in spheres

Example

$\pi_3(S^2) = \mathbb{Z}$, generated by the Hopf fibration $S^3 \to S^2$, which corresponds to S^1 embedded in S^3 as usual but with a twisted trivialization of its normal bundle.

Bad news: $\pi_{n+k}(S^n) \simeq \Omega_k^{fr}(S^{n+k})$ is extremely complicated! Going stable helps, but $\pi_*^s = \mathbf{colim}_n \, \pi_{n+k} S^k$ is still daunting.

Coping mechanism:

$$\pi_*^s \simeq \Omega_*^{fr} \to \Omega_*^U, \quad \Omega_*^{fr} \to \Omega_*^O, \quad \Omega_*^{fr} \to \Omega_*^{SO}, \quad \Omega_*^{fr} \to \Omega_*^{Spin}, \quad \Omega_*^{fr} \to \Omega_*^G$$

Genera

Ω_*^G: bordism ring of manifolds with G-structure on their stable normal bundle.

Definition

An R-valued genus is a (graded) ring homomorphism $\Omega_*^G \to R$.
A complex genus is a map $\phi : \Omega^U \to R$.

Example (Cardinality for zero-dimensional manifolds)

$\Omega_*^O \to \mathbb{Z}/2$ $\qquad$ $\Omega_*^O = \mathbb{Z}/2[x_i]_{i \neq 2^j}$

$\Omega_*^{SO} \to \mathbb{Z}$ $\qquad$ $\Omega_*^{SO} \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \dots]$

sending positive-dimensional manifolds to zero.

More examples

Example (Todd genus)

The Todd genus Td has $R = \mathbb{Z}[u]$, with $\deg u = 2$.
$\Omega_*^U \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^1, \mathbb{C}P^2, \dots]$, and $Td(\mathbb{C}P^k) = u^k$.

Example ($\hat{A}$-genus)

Maps a spin manifold M^n to the index of its Dirac operator.

- $n \equiv 0(8)$, $\hat{A}(M^n)$ is an integer
- $n \equiv 4(8)$, $\hat{A}(M^n)$ is an *even* integer
- $n \equiv 1(8)$, $\hat{A}(M^n)$ is defined mod 2

Example (Witten genus)

Assigns a modular form to a (compact, oriented, smooth) spin manifold with vanishing first Pontryagin class.

Spectra

Cobordism in families $\rightsquigarrow$ cohomology theory

$$\begin{aligned} E : (Spaces)^{op} &\longrightarrow (Gr.Modules) \\ X &\longmapsto E^*(X) \end{aligned}$$

such that

- $f \simeq g \implies E^*(f) = E^*(g)$
- Mayer-Vietoris sequences

Cohomology theories are represented by spectra.

Example

- Cobordism Ω^G
- Singular cohomology theory $H\mathbb{Z}$, $H\mathbb{Z}/p$...
- Complex and real K-theory, K and KO

Genera again

Definition

An genus is a map of multiplicative cohomology theories (ring spectra)

$$\Omega^G \to E.$$

Also known as a G-orientation for E, because it gives a theory of Thom classes for bundles with G-structure.

Example

Cardinality	$\Omega^O \to H\mathbb{Z}/2$
	$\Omega^{SO} \to H\mathbb{Z}$
Todd genus	$\Omega^U \to K$
$\hat{A}-$genus	$\Omega^{Spin} \to KO$
Witten genus	$\Omega^{String} \to TMF.$

Formal group laws

Theorem (Quillen)

Ω_*^U *carries a universal* *formal group law.*

$\Omega^U \to E$ and $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty \rightsquigarrow$

$$E^0[[x]] \cong E^0(\mathbb{C}P^\infty) \longrightarrow E^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^0[[x,y]]$$

$$x \longmapsto F(x,y) = x +_F y$$

Definition/Properties

- $x +_F 0 = x = 0 +_F x$
- $(x +_F y) +_F z = x +_F (y +_F z)$
- $x +_F y = y +_F x$

Examples

Formal completions of algebraic groups of dimension 1

- Additive group $\mathbb{G}_a$
  - $F(x, y) = x + y$
  - $H\mathbb{Z}$, $H\mathbb{Z}/p$
- Multiplicative group $\mathbb{G}_m$
  - $F(x, y) = x + y + uxy$, where u is a unit
  - K, $u = \pm 1$ or the Bott class
- Elliptic curves
  - next time!

Logarithms

Rationally, every formal group law is isomorphic to the additive!

- $\exists f(x) \in R[[x]] \otimes \mathbb{Q}$, with $f'(0) = x$, such that $f(x +_F y) = f(x) + f(y)$

Such an f is called a logarithm for F.

Example

$\log_{\hat{\mathbb{G}}_m} = \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

Height of a formal group law

Pick a prime p, an $\mathbb{F}_p$-algebra R, and a formal group law F over R. The p-series of F is $[p]x = [p]_F(x) = \underbrace{x +_F \cdots +_F x}_{p \text{ times}}$.

Example

$[p]_{\hat{\mathbb{G}}_a}(x) = 0$
$[p]_{\hat{\mathbb{G}}_m}(x) = u^{p-1}x^p\phi$, where $\phi \in (R[[x]])^\times$

Definition

F has height n if $[p]_F(x)$ has leading term ax^{p^n}. If $[p]_F(x) = 0$, the height of F is ∞.

For any $n \geq 1$, there is a formal group law of height n, and any two are isomorphic (after an étale extension).

Chromatic homotopy theory

Heights induce the chromatic filtration on π_*^s.
If E is a complex oriented cohomology theory with a formal group law of height n, then

$$\pi_*^s \to \Omega_*^U \to E^{-*}$$

captures info about filtration at most n.

- $\pi_*^s \to H\mathbb{Z}^{-*} = \mathbb{Z}$
- $\pi_*^s \to K^{-*}$ height one
- KO is not complex orientable, but $\pi_*^s \to \Omega_*^{Spin} \to KO^{-*}$ is the best approximation we have of height one data

Closing remarks

- There is a way to translate and improve a lot of this into derived algebraic geometry
- By work of Lurie, in finding structured cohomology theories with higher height fgl's, p-divisible groups are needed (more general than formal groups)
  - moduli of elliptic curves (height at most 2) $\rightsquigarrow$ TMF
  - moduli of some higher dimensional abelian varieties (arbitrary height) $\rightsquigarrow$ TAF (Behrens-Lawson)

To be continued with

- Elliptic curves for the homotopy theorist
- Topological modular forms of various sorts
- A lot of duality