Duality and Topological Modular Forms

Vesna Stojanoska

MIT

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Definition

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Dualizing Modules in Algebra

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Example

$\mathbb{Z}$ is a dualizing $\mathbb{Z}$-module.
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**Example**

For the projective line $f : \mathbb{P}^1 \to \text{Spec } \mathbb{Z}$, the sheaf of Kahler differentials $\Omega_{\mathbb{P}^1}$ is a dualizing module.
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Vesna Stojanoska

Duality and $Tmf$
Definition

For a ring spectrum \( A \), a \textit{dualizing} \( A \)-\textit{module} is an \( A \)-module \( K \) such that

(i) the double duality map \( A \to F_A(K, K) \) is an equivalence,

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\( K \) has a finite injective dimension, i.e. there exists an integer \( n \) such that if \( M \) is an \( A \)-module with \( \pi_i M = 0 \) for \( i > n \), then \( \pi_i F_A(M, K) = 0 \) for \( i < 0 \).

Example

The sphere spectrum \( S \) is \textit{not} a dualizing module over itself!
Brown-Comenetz spectrum $\mathbb{Q}/\mathbb{Z}$
Anderson Duality

Brown-Comenetz spectrum $l_{\mathbb{Q}/\mathbb{Z}}$

\[ X \mapsto \text{Hom}_{\mathbb{Z}}(\pi_* X, \mathbb{Q}/\mathbb{Z}) \]
Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

\[ X \mapsto \text{Hom}_{\mathbb{Z}}(\pi_* X, \mathbb{Q}/\mathbb{Z}) \]

Rational Eilenberg-MacLane spectrum $H\mathbb{Q}$
Anderson Duality

Brown-Comenetz spectrum $I_{Q/Z}$

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Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

$$X \mapsto \text{Hom}_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Q}/\mathbb{Z})$$

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Anderson spectrum $I_{\mathbb{Z}}$

fiber sequence $I_{\mathbb{Z}} \to H\mathbb{Q} \to I_{\mathbb{Q}/\mathbb{Z}}$
Anderson Duality

Brown-Comenetz spectrum $I_{\mathbb{Q}/\mathbb{Z}}$

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Anderson spectrum $I_{\mathbb{Z}}$

fiber sequence $I_{\mathbb{Z}} \rightarrow H\mathbb{Q} \rightarrow I_{\mathbb{Q}/\mathbb{Z}}$

Example

The Anderson spectrum $I_{\mathbb{Z}}$ is a dualizing $S$-module.
Eilenberg-MacLane spectra $HM$, $M$ finite or free
Self-dual Spectra

- Eilenberg-MacLane spectra $HM$, $M$ finite or free
- Complex and real $K$-theory
Eilenberg-MacLane spectra $HM$, $M$ finite or free

Complex and real $K$-theory

$Tmf$, $Tmf(p)$
Self-dual Spectra

- Eilenberg-MacLane spectra $HM$, $M$ finite or free
- Complex and real $K$-theory
- $Tmf$, $Tmf(p)$
- ...?
Duality for $K$-theory

$I_{\mathbb{Z}} K \cong K$
Duality for $K$-theory

$I_\mathbb{Z} K \cong K$

$K^{C_2}$ and $KO \cong K^{hC_2}$
\[ I_\mathbb{Z} K \cong K \]

\[ K \overset{\mathbb{C}_2}{\cong} \text{and } KO \cong K^{h\mathbb{C}_2} \]

\[ (BC_2, K) \overset{f}{\rightarrow} \text{Spec } S \]
Duality for $K$-theory

$I_{\mathbb{Z}}K \simeq K$

$K^{C_2}$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \overset{f}{\rightarrow} \text{Spec } S$

derived stack, $R\Gamma = (\_)^{hC_2}$
Duality for $K$-theory

\[ \mathbb{I}_\mathbb{Z}K \simeq K \]

\[ K^\mathbb{C}_2 \text{ and } KO \simeq K^{hC_2} \]

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\[ \mathbb{I}_\mathbb{Z}KO = F(R\Gamma K, \mathbb{I}_\mathbb{Z}) \]
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norm $K_{hC_2} \to K^{hC_2}$ is an equivalence
Duality for $K$-theory

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$(K^{tC_2} \simeq *)$
Duality for $K$-theory

\[ \mathbb{I}_\mathbb{Z} K \cong K \]

$K \circlearrowleft C_2$ and $KO \cong K^{hC_2}$

\[(BC_2, K) \xrightarrow{f} \text{Spec } S\]

derived stack, $R\Gamma = (-)^{hC_2}$

\[ \mathbb{I}_\mathbb{Z} KO = F(R\Gamma K, \mathbb{I}_\mathbb{Z}) \cong (\mathbb{I}_\mathbb{Z} K)^{hC_2} \]

norm $K_{hC_2} \to K^{hC_2}$ is an equivalence

\[ (K^{tC_2} \cong *) \]
Duality for $K$-theory

$I_{\mathbb{Z}}K \simeq K$

$K \langle C_2 \rangle$ and $KO \simeq K^{hC_2}$

$(BC_2, K) \xrightarrow{f} \text{Spec } S$

derived stack, $R\Gamma = ( - )^{hC_2}$

$I_{\mathbb{Z}}KO = F(R\Gamma K, I_{\mathbb{Z}}) \simeq (I_{\mathbb{Z}}K)^{hC_2} = \Sigma^4 KO$

norm $K_{hC_2} \to K^{hC_2}$ is an equivalence

$(K^{tC_2} \simeq *)$
Duality for $K$-theory

$I_{\mathbb{Z}}K \cong K$

$K \otimes_{C_2} K \otimes C_2$ and $KO \cong K^{hC_2}$

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derived stack, $R\Gamma = (-)^{hC_2}$

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Warning Does not work for trivial action, or for $K_G$. 
$Tmf$ and level structures

Derived stack $(\mathcal{M}, \mathcal{O}^{\text{top}}) \xrightarrow{f} \text{Spec } S$
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(compactified moduli stack of elliptic curves)
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$R\Gamma \mathcal{O}^{top} = Tmf$
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$\mathcal{M}(p) \to \mathcal{M}[1/p]$
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an \(SL_2(\mathbb{Z}/p)\)-cover, ramified at infinity
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Construct $\mathcal{O}(p)^{top}$ and $Tmf(p) = R\Gamma \mathcal{O}(p)^{top}$
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Cusps: $\coprod_{SL_2(\mathbb{Z}/p)/U} \text{Spf } \mathbb{Z}[[q^{1/p}]] \to \text{Spf } \mathbb{Z}[[q]]$
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**Construct \(\mathcal{O}(p)^{\text{top}}\) and \(\text{Tmf}(p) = R\Gamma \mathcal{O}(p)^{\text{top}}\)**

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\(K[[q^{1/p}]]\) has \(U\)-action \((\text{Cooke's obstruction theory})\)
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$\mathcal{O}(p)^{top}$ on the cusps is $SL_2(\mathbb{Z}/p)_+ \bigwedge U K[[q^{1/p}]]$
The construction implies Descent
Duality for $Tmf(p)$

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**Descent**

\[ Tmf[1/p] \simeq Tmf(p)^{hSL_2(\mathbb{Z}/p)} \]
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\[
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\[Tmf[1/p] \simeq Tmf(p)^{h\text{SL}_2(\mathbb{Z}/p)}\]

For \(p = 2, p = 3\), \(M(p)\) is a weighted projective line
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Duality for $Tmf(p)$

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Serre duality implies

**Theorem (S.)**

\[ l_{\mathbb{Z}} Tmf(2) = \Sigma^9 Tmf(2) \quad l_{\mathbb{Z}} Tmf(3) = \Sigma^5 Tmf(3) \]
Theorem (S.)

The Tate spectra $Tmf(2)^{t\text{SL}_2(\mathbb{Z}/2)}$, $Tmf(3)^{t\text{SL}_2(\mathbb{Z}/3)}$ are contractible.
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The Anderson dual of $Tmf$ is $\Sigma^{21} Tmf$. 
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Theorem (S.)

The Anderson dual of $Tmf$ is $\Sigma^{21} Tmf$.

Sheafification: Indicates that $\Sigma^{21} O^{top}$ is a dualizing $O^{top}$-module, in contrast with the ordinary geometry.
Thank you!