MATH 402 Worksheet 7

Friday 10/14/16

(1) The goal of this exercise is to give a model-independent proof of the *Fundamental theorem of parallels* in hyperbolic geometry, which states:

Given a hyperbolic line l and a point P not on l, there are exactly two lines m, n, through P and parallel to l such that

(i) *Every line through P lying within the angle made by m and n for which l is interior intersects l, while all other lines through P are parallel to l.*

(ii) *m and n make equal acute angles with the perpendicular to l through P.*

To prove this theorem, proceed as follows:

(a) Let Q be the intersection point of l and the perpendicular to l through P. Consider all angles (of at most 180°) of the form $\angle QPX$, for some ray $\overrightarrow{PX}$. These angles will fall into two sets: those with $\overrightarrow{PX}$ intersecting l and those with $\overrightarrow{PX}$ not intersecting l. There is a ray $\overrightarrow{PC}$ separating these, and it does not intersect l. Why?

(b) Reflect $\overrightarrow{PC}$ across the line PQ. What can you say about the resulting ray, and the angles formed?

(c) Could $\angle QPC$ be bigger than 90°? Prove that if it is so, there is a triangle with angle sum greater than 180°.

(d) Could $\angle QPC$ be equal to 90°? Prove that it cannot by contradiction; this time use the axiom for hyperbolic parallels and the preceding results in this exercise.

In the theorem above, m and n are called limiting parallels *to l; all other parallels are called* divergent *or* ultraparallels*.*

(2) Show that if m is a limiting parallel to l through $P \in m$, then m is a limiting parallel to l through any other point $P' \in m$.

Use the following strategy:

(a) Drop perpendiculars PQ and $P'Q'$ to l, with $Q, Q' \in l$. Prove that $PQ||P'Q'$.

(b) Consider rays $P'S$ for points S in various locations. Assuming that $P'S$ intersects l, show that also PS intersects l. You will need to use Pasch's axiom.

Bonus:

(3) Show that the property of being a limiting parallel is symmetric, i.e. if m is a limiting parallel to l, then l is a limiting parallel to m.