MATH 402 Worksheet 4

Friday 9/16/16

(1) Last time we proved the following fact: If $\triangle ABC$ is congruent to $\triangle XYZ$, then there is an isometry f such that $f(\triangle ABC) = \triangle XYZ$, and such that f is a composition of at most three reflections. Use this fact to prove that any isometry of the plane can be written as the composition of at most three reflections.

Now we define the following kinds of isometries:

- A composition of two reflections $r_{l_2} \circ r_{l_1}$ is called a *translation* if the lines of reflection l_1 and l_2 are parallel to each other.
- A composition of two reflections $r_{l_2} \circ r_{l_1}$ is called a *rotation* if the lines of reflection l_1 and l_2 are not parallel to each other.

All of the following claims work in neutral geometry as well, so work without using the parallel postulate.

(2) Prove that a translation has no fixed points.

(3) Prove that a rotation which is not the identity has a single fixed point. That point is called the center of rotation.

(4) Conversely, suppose f is an isometry with a unique fixed point O. Prove that f must be a rotation. It might be helpful to first prove these facts about reflections and rotations:

(a) Let l_1 and l_2 be lines that intersect at a point O, and let P_1 and P_2 be points on l_1 and l_2 respectively. Let m be the angle bisector of $\angle P_1OP_2$. Then $r_m(l_1) = l_2$, and moreover,

$$r_m \circ r_{l_1} = r_{l_2} \circ r_m.$$

(b) Let l, m, n be three lines that intersect at a point O. Then $r_l \circ r_m \circ r_n$ is a reflection about a line p through O.