MATH 402 Worksheet 2

Friday 9/2/16

(1) Prove the *vertical angle theorem* within Hilbert's axiomatic system. Namely, let l, m be lines intersecting at P. Show that the vertical angles formed by l and m are equal to each other.

(2) The following theorem is called the *exterior angle theorem*: Given a triangle $\triangle ABC$, extend one of its sides, for example $\overline{AC}$ to D. Then, the exterior angle produced (i.e. $\angle DAB$) is greater than either of the two opposite interior angles (i.e. $\angle ABC$ and $\angle ACB$).

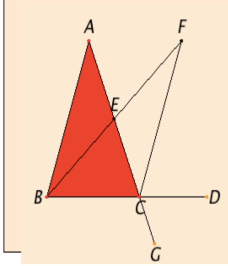
Euclid gave the following proof:

Proposition 16

In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.

Let *ABC* be a triangle, and let one side of it *BC* be produced to *D*.

I say that the exterior angle *ACD* is greater than either of the interior and opposite angles *CBA* and *BAC*.



Bisect *AC* at *E*. Join *BE*, and produce it in a straight line to *F*.

Make *EF* equal to *BE*, join *FC*, and draw *AC* through to *G*.

Since *AE* equals *EC*, and *BE* equals *EF*, therefore the two sides *AE* and *EB* equal the two sides *CE* and *EF* respectively, and the angle *AEB* equals the angle *FEC*, for they are vertical angles. Therefore the base *AB* equals the base *FC*, the triangle *ABE* equals the triangle *CFE*, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides. Therefore the angle *BAE* equals the angle *ECF*.

But the angle *ECD* is greater than the angle *ECF*, therefore the angle *ACD* is greater than the angle *BAE*.

Similarly, if *BC* is bisected, then the angle *BCG*, that is, the angle *ACD*, can also be proved to be greater than the angle *ABC*.

Therefore *in any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles*.

Why is this proof flawed within the Euclidean system? What can be done to salvage it in Hilbert's system?

(3) Prove the following converse to the parallel postulate (without using the parallel postulate): Let l, m be two lines, and n a third line intersecting both l, m, such that the alternate interior angles formed are congruent. Then l and m are parallel.

(4) Formally define congruence of triangles.

(5) Euclid proved the side-angle-side (SAS) congruence, but there are issues with his proof. In contrast, Hilbert assumed SAS as an axiom. Using Hilbert's axioms, state and prove the other three congruence rules: ASA, AAS, SSS.