MATH 402 Practice for first exam

Topics: material covered in class (through lectures, worksheets, projects, or homework) by the exam date. This corresponds to sections 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, Appendices A & D, 2.1, 2.2, 2.3, 2.5, 2.6, 2.7, 3.1, 3.2, 3.4, 5.1, 5.2, 5.3, 5.4, 5.6, 5.7.

1. All problems from homework, worksheet, and projects, as well as other problems from the book.

2. Definitions and statements

   (a) Define parallel and perpendicular

   (b) State Euclid's 5th postulate, and Playfair's postulate

   (c) Define congruent triangles, similar triangles

   (d) State Pasch's axiom

   (e) State the supplementary angle theorem, vertical angle theorem, exterior angle theorem

   (f) State SAS/SSS/ASA congruence rules, and SSS/AAA similarity rules

   (g) Define neutral geometry

   (h) Define a chord, diameter, central angle, inscribed angle, tangent line to a circle

   (i) Define the inverse of a point with respect to a circle

   (j) Define an isometry of the plane

   (k) State the basic properties of isometries (Theorem 5.1)

   (l) Define a fixed point of an isometry

   (m) Define reflections, rotations, translations, glide reflections

   (n) Define line of reflection, angle of rotation, vector of translation

   (o) How to represent reflections, rotations, and translations with matrices

3. Proofs

   (a) Prove the vertical angle theorem.

   (b) Prove the exterior angle theorem.

   (c) Let $\mathcal{C}$ be a circle with origin O. Prove that there isn't a circle $\mathcal{D}$ which goes through O and such that $\mathcal{C}$ and $\mathcal{D}$ are orthogonal.

   (d) Prove that Playfair's postulate implies the following statement: If l_1 and l_2 are two unequal parallel lines, and m is another line which intersects l_1 (but is not equal to l_1), then m also intersects l_2.

   (e) Given Hilbert's axioms, prove SSS and ASA.

   (f) Consider the axiomatic system defined by the following. The undefined terms are points, and a line is defined as a set of points. The axioms are:

      i. There are exactly four points.

      ii. There are exactly four lines.

iii. Given any two different points there exists at least one line that contains them.

- I claim that this system is consistent. Give a model.
- Show that each of these three axioms is independent from the others.
- Is the system complete? Why or why not?
- Is the following statement true or false for the system: Every line contains at most two points. Justify your answer.
- From the given system S form a new one T which is "dual," i.e. points in T are lines in S, and lines in T are points in S. Does T satisfy the same axioms as S? If the answer is yes, prove your claim by proving each axiom holds for T. If the answer is no, prove that one of the axioms for S (which one?) does not hold.

(g) Let l, m, n be three lines, and r_l, r_m, r_n the associated reflections.

- If the intersection $l \cap m \cap n$ is a point O, what kind of isometry is $r_n \circ r_m \circ r_l$? Prove your answer.
- If l, m, n are all parallel to each other, what kind of isometry is $r_n \circ r_m \circ r_l$? Prove your answer.

(h) Let f be any isometry, and r_l reflection about a line l. Show that the composition $f \circ r_l \circ f^{-1}$ is reflection about the line $f(l)$.

(i) Let f be any isometry, and $T_{\vec{v}}$ translation by vector $\vec{v}$. Show that the composition $f \circ T_{\vec{v}} \circ f^{-1}$ is translation by vector $f(\vec{v})$.

(j) Suppose l and m are lines that intersect at a point O. What is the angle of rotation for the composition $r_m \circ r_l$? Prove your answer.

(k) In general, know how to determine compositions of various isometries. When the setup is very complicated, using matrix representations of isometries will be helpful.

4. True or false

(a) Euclid's 5th postulate is inconsistent with the other four.

(b) Euclid's 5th postulate is independent from the other four.

(c) In Euclidean geometry, a line and a circle can have exactly one point of intersection.

(d) The parallel postulate is not needed to prove that the sum of angles in a triangle is $180°$.

(e) The image of a triangle under an isometry gives a similar triangle.

(f) The composition of two rotations can be a translation.

(g) The composition of three reflections can be a rotation.