Name:

netid.:

Math 402: Exam 1

Fall semester 2016

- Do not forget to write your name and netid on top of this page.
- No notes, books, calculators, or other exam aids are allowed.
- Turn your cell phones off and put them away. **No use of cell phones** or other communication devices during the exam is allowed.
- Write your answers **clearly and fully** on the sheets provided. If you need additional paper, raise your hand.
- **Do not tear pages** off of this exam. Doing so will be considered cheating.
- The exam consists of 6 problems and 9 pages. Check that your exam is complete.
- You have **50 minutes** to complete the exam.

Good luck!!

Problem	1	2	3	4	5	6	Σ
Total possible	12	25	13	20	10	20	100
Your points							

Problem 1: (8 + 4 = 12 Points)
This problem concerns the two axiomatic systems, $\mathcal{A}$ and $\mathcal{B}$ below. For both of them, the undefined terms are points, and a line is defined to be a set of points.

Axiomatic system $\mathcal{A}$

Axiom A1: There are two different points which have two lines in common.

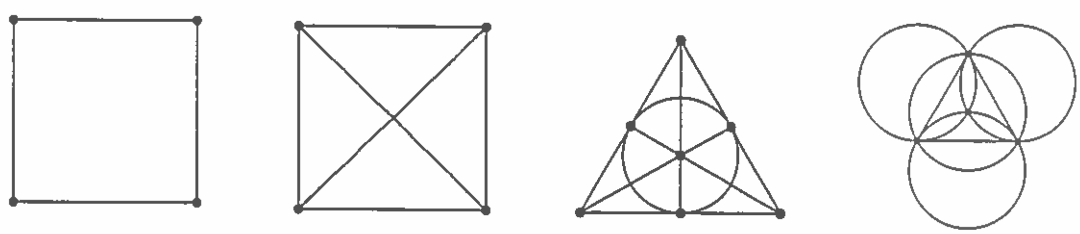
Axiom A2: There are two different lines which have two points in common.

Axiomatic system $\mathcal{B}$

Axiom B1: Any two different points have exactly one line in common.

Axiom B2: Any two different lines have exactly one point in common.

(a) The following are depictions of potential models; points are depicted as points, and lines as straight line segments or circles. Which of the four depictions are models for which system (if any)? Write YES or NO in the corresponding box of the table.



(a) Model I (b) Model II (c) Model III (d) Model IV

	Model I	Model II	Model III	Model IV
System $\mathcal{A}$	NO	NO	NO	YES
System $\mathcal{B}$	NO	NO	YES	NO

(b) Is the system $\mathcal{B}$ complete? Why or why not?

No, because we can add an independent axiom & keep the system consistent.

eg. $\exists$ exactly one point $\rightsquigarrow$ •

$\exists$ more than one point $\rightsquigarrow$ •—• or model III

Problem 2: (8 + 7 + 10 = 25 Points)

(a) State Pasch's axiom.

Let A, B, C be non-collinear points, and let m be a line not containing any of A, B, C. If m intersects the segment $\overline{AB}$, then it must also intersect exactly one of $\overline{BC}$ or $\overline{AC}$.

(b) Suppose l is a line, and P, Q two points not on l. Define what it means for the points P, Q to be *on the same side* of l, and what it means for P, Q to be *on opposite sides* of l.

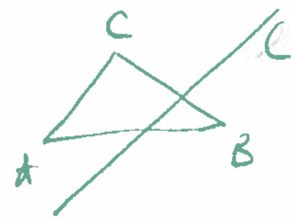
P, Q are on the same side of ℓ if ℓ contains no points of the segment $\overline{PQ}$; they are on opposite sides of ℓ if ℓ intersects $\overline{PQ}$.

(c) Given a line l, suppose that A, B, C are three points none of which is on l, and such that A and B are on opposite sides of l, and B and C are on opposite sides of l. Prove that A and C must be on the same side of l.

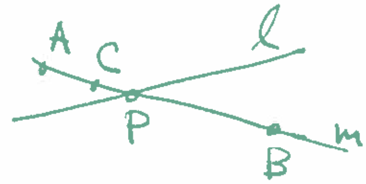
By assumption, ℓ intersects $\overline{AB}$ as well as $\overline{BC}$.

There are two cases:

1) A, B, C are non-collinear. Since $A, B, C \notin \ell$, Pasch's axiom applies to say ℓ cannot intersect $\overline{AC}$, i.e. A & C are on the same side of ℓ.



2) A, B, C are collinear, lying on line m.



~~Let~~ Then m & ℓ intersect (at a point we call P), since $\overline{AB}$ & ℓ intersect.

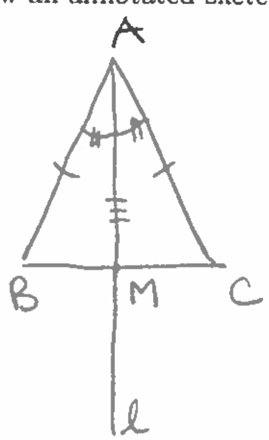
By assumption, P is between A & B, and between B and C. This means that A is not on the ray $\overrightarrow{PB}$ and likewise C is not on the ray $\overrightarrow{PB}$. But all points on m which are not on $\overrightarrow{PB}$ are on the same side of ℓ.

Problem 3: (5 + 8 = 13 Points)
The following theorem holds in Euclidean geometry:

In an isosceles triangle, the angle bisector of the summit is the perpendicular bisector of the base.

Here is a standard argument for it, using the SAS congruence rule, which is an axiom in Hilbert's system for Euclidean geometry.
Let $\triangle ABC$ be an isosceles triangle with base $\overline{BC}$. Let l be the angle bisector of $\angle BAC$, intersecting $\overline{BC}$ at point M. We need to show that l is perpendicular to $\overline{BC}$ and $\overline{BM} \cong \overline{CM}$. We have $\overline{AB} \cong \overline{AC}$, $\angle BAM \cong \angle CAM$, and $\overline{AM} \cong \overline{AM}$, hence by SAS we have a congruence of triangles $\triangle ABM \cong \triangle ACM$. This implies $\overline{BM} \cong \overline{CM}$, so l bisects $\overline{BC}$. Moreover, $\angle AMB \cong \angle AMC$, but they are also supplementary, so they have to be right angles, i.e. l is perpendicular to $\overline{BC}$.

(a) Draw an annotated sketch to accompany the above argument.



(b) Does the above argument use anything other than axioms and definitions in Hilbert's system? Explain.

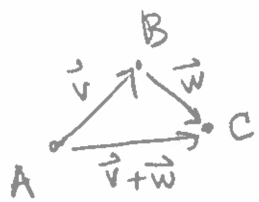
Yes, it uses that l will intersect the segment $\overline{BC}$.

Problem 4: (10 + 10 = 20 Points)

(a) Let f be an isometry of the Euclidean plane. Show that f is additive for vectors, i.e. for $\vec{v}, \vec{w}$ arbitrary vectors prove that

$$f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w}).$$

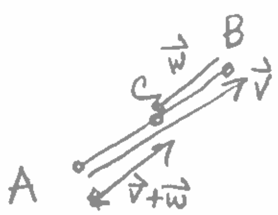
Let A, B, C be points such that $\vec{v} = \overrightarrow{AB}$, $\vec{w} = \overrightarrow{BC}$, hence $\vec{v} + \vec{w} = \overrightarrow{AC}$.



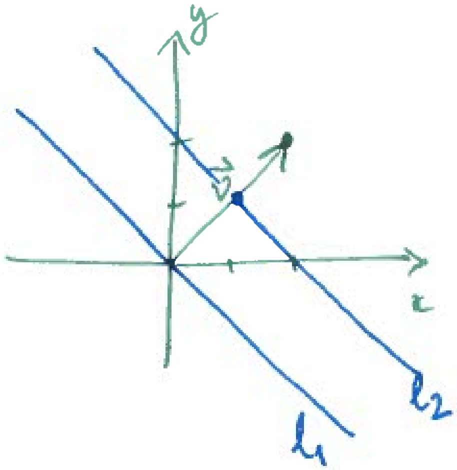
then $f(\vec{v}+\vec{w}) = f(\overrightarrow{AC}) = \overrightarrow{f(A)f(C)} =$

$= \overrightarrow{f(A)f(B)} + \overrightarrow{f(B)f(C)} = f(\overrightarrow{AB}) + f(\overrightarrow{BC}) =$

$= f(\vec{v}) + f(\vec{w})$.



(b) Let $\vec{v}$ be the vector $(2,2)$ in a chosen coordinate system for the Euclidean plane, and let $T_{\vec{v}}$ be the translation of the Euclidean plane by $\vec{v} = (2,2)$. Precisely describe two lines l_1, l_2, such that $T_{\vec{v}}$ can be written as the composition $r_{l_2} \circ r_{l_1}$ of the two associated reflections?

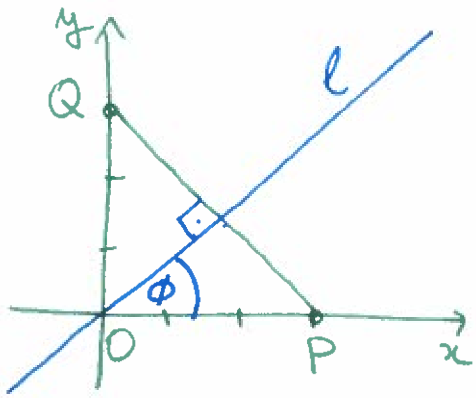


Any two $l_1 \parallel l_2$ s.t. the vector from l_1 to l_2 is $\frac{1}{2}\vec{v}$ will do. (so, $\vec{v} \perp l_1$, $\vec{v} \perp l_2$).

For example l_1 : $x+y=0$

l_2 : $x+y=2$

Problem 5: (10 Points)
We are given a coordinate system for Euclidean geometry. Consider the points $P = (3,0)$ and $Q = (0,3)$. Precisely describe (as a matrix, or a string of matrices to be multiplied) a reflection r_l such that $r_l(P) = Q$.



ℓ must be the perp. bisector of $\overline{PQ}$, which is the line $x=y$.
the angle ℓ makes w/ x-axis is $45°$.

- Rotation $R_{0,45°} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$; then

$$r_\ell = R_{0,45°} \circ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \circ R_{0,-45°}.$$

Problem 6: (10 × 2 = 20 Points)
Are the following statements true or false? Circle *neatly* the correct answer.
No partial credit will be given.

(a)	Euclid's 5th postulate is inconsistent with the other four.	True	False
(b)	The congruence rule ASA is a theorem in Hilbert's system.	True	False
(c)	In neutral geometry, two different lines could intersect at more than one point.	True	False
(d)	The measure of an inscribed angle is twice that of its intercepted central angle.	True	False
(e)	Every Euclidean triangle is inscribed in a unique circle.	True	False
(f)	Let P be a point in the Euclidean plane, lying on a circle c. There are exactly two tangents to c going through P.	True	False
(g)	The composition of four reflections could be a glide reflection.	True	False
(h)	The composition of two different reflections could have two different fixed points.	True	False
(i)	An isometry with no fixed points is a translation.	True	False
(j)	If f is an isometry which has three different invariant lines, then f is the identity.	True	False