MATH 402 Non-Euclidean Geometry

Worksheet 7 on hyperbolic distance and reflections

Friday 11/13.

Recall from last time that for two points z_0, z_1 in the Poincaré disk, the hyperbolic distance between them is given by

$$d_p(z_0, z_1) = |\ln(z_0, z_1, w_1, w_0)|,$$

where w_0, w_1 are the ideal points of the hyperbolic line through z_0, z_1.

(a). Suppose z is a point in the Poincaré disk. Prove that the hyperbolic distance between 0 and z is given by

$$d_p(0, z) = \ln\left(\frac{1+|z|}{1-|z|}\right).$$

(b). Now let z_0, z_1 be any two points in the Poincaré disk. Your goal is to find a formula for $d_p(z_0, z_1)$ that only involves z_0, z_1 (and not w_0, w_1).

Hints: *First, find an isometry g of the Poincaré disk which moves z_1 to 0, and then use the previous exercise.*

Again, recall from last time that hyperbolic reflections across a hyperbolic line $\mathcal{D}$ are of the following form:

- If $\mathcal{D}$ is a diameter of the unit circle, then hyperbolic reflection $\rho_{\mathcal{D}}$ about $\mathcal{D}$ is the same as Euclidean reflection about $\mathcal{D}$, and
- If $\mathcal{D}$ is a Euclidean circle centered at a (with $|a| > 1$), then

$$\rho_{\mathcal{D}}(z) = \frac{|a|^2 - 1}{\overline{z} - \overline{a}} + a.$$

(c). Suppose $\mathcal{D}$ is a diameter of the unit circle which intersects the unit circle at a. Write down a formula for $\rho_{\mathcal{D}}(z)$.

Hints: *First, determine ρ if $a = \pm 1$. Then use a rotation to reduce the problem to the case when $a = \pm 1$.*

(d). Just as in Euclidean geometry, given two points z_0, z_1 in the Poincaré disk, there exists a unique reflection ρ such that $\rho(z_0) = z_1$. Determine the equation for $\rho(z)$ in the following cases:

(i) $z_1 = \frac{1+i}{2}$ and $z_0 = \frac{-1-i}{2}$

(ii) $z_1 = 0$

(iii) $z_1 = \frac{z_0}{2}$

(iv) Can you do this for general z_0, z_1?