MATH 402 Non-Euclidean Geometry

Worksheet 6 on Cross Ratios

Friday 11/06.

The cross-ratio of four complex numbers z_0, z_1, z_2, z_3 is defined as

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3} \frac{z_1 - z_3}{z_1 - z_2}.$$

(a). Suppose z_1, z_2, z_3 are three distinct complex numbers, and let f be a Möbius transformation. Prove that for any z,

$$(f(z), f(z_1), f(z_2), f(z_3)) = (z, z_1, z_2, z_3).$$

Hint: *For the function $g(z) = (z, z_1, z_2, z_3)$, analyze the effect of $g \circ f^{-1}$ on $f(z_1), f(z_2), f(z_3)$. How does that compare to the function $h(z) = (z, f(z_1), f(z_2), f(z_3))$?*

(b). In this part, you will the follow the outline steps to prove the following theorem: Let z_0, z_1, z_2, z_3 be four distinct points. Then the cross ratio (z_0, z_1, z_2, z_3) is a real number if and only if the four points lie on a cline.

- To start the proof, you use that three distinct points define a unique cline. So, z_1, z_2, z_3 define a unique cline, and the question is how to tell if z_0 is on it or not. What could the equation for a cline through z_1, z_2, z_3 look like?
- Show that if z_0 satisfies such an equation, then the cross ratio is real.
- Now you move on to the converse. Study the function

  $$f(z) = (z, z_1, z_2, z_3).$$

  How can you tell if $f(z)$ is real?