MATH 402 Non-Euclidean Geometry
Worksheet 5 on Hyperbolic area

Friday 10/30.

We say that two hyperbolic polygons are *scissor congruent* (the book calls them "equivalent") if one of them can be cut into pieces (using hyperbolic scissors) and then glued without overlaps to give the second polygon.

(a). Of course, the analogous notion exists in Euclidean geometry as well. Prove that every Euclidean triangle is scissor congruent to a rectangle.

(b). What is the area of a Euclidean rectangle?

(c). Use this to show that the area of a Euclidean triangle is one half of the base times the height.

In hyperbolic geometry there are no rectangles, so we need to use a more general, axiomatic approach to are.

The following are the area axioms:

Area 1 If the points A, B, C are not collinear, then the area of the triangle $a(\triangle ABC)$ is positive.

Area 2 Two polygons which are scissor congruent have the same area.

Area 3 The area of the union of disjoint sets X, Y is the sum of areas $a(X)+a(Y)$.

(d). The goal of this exercise is to prove the following theorem.

Theorem. Suppose $\triangle ABC$ and $\triangle A'B'C'$ are hyperbolic triangles such that $\overline{AB} = \overline{A'B'}$, and their defects are equal. Then $\triangle ABC$ is scissor congruent to $\triangle A'B'C'$, so they must have the same area.

To prove this, you will show that both triangles are scissor congruent to the same Saccheri quadrilateral. Here are some detailed hints:

(i) Draw a line l through the midpoints M and N of $\overline{AC}$ and $\overline{BC}$.

(ii) Drop perpendiculars to l through A, B, C.

(iii) Find a Saccheri quadrilateral $\mathcal{Q}$ with base on l.

(iv) Show that some of the resulting triangles are congruent to each other.

(v) Deduce that $\triangle ABC$ is scissor congruent to the Saccheri quadrilateral.

(vi) How do the summit angles of $\mathcal{Q}$ compare to the angles of the triangle $\triangle ABC$?

(vii) To finish the argument, use the following fact (Exercise 7.5.4): Two Saccheri quadrilaterals with congruent summits and equal angles must be congruent.

(e). Now, try to prove the following.

Theorem. Two triangles $\triangle ABC$ and $\triangle A'B'C'$ with the same defect are scissor congruent even if they don't have a pair of sides with equal length.

Here are some hints:

(i) Start with the setup as in the previous theorem.

(ii) Construct an intermediate triangle $\triangle ABC''$ which has the following properties:

- Its defect is the same as that of the given two triangles, and
- It shares one side with $\triangle ABC$ and another side with $\triangle A'B'C'$.
- Use the previous theorem.

(f). In fact, the theorem above has a converse. Try to prove it without any hints. Then generalize to polygons in neutral geometry.

Theorem. If two triangles $\triangle ABC$ and $\triangle A'B'C'$ are scissor congruent, then they have the same defect.