MATH 402 Non-Euclidean Geometry

Worksheet 4 on Euclidean transformations

Friday 10/9.

1. Let $\triangle ABC$ and $\triangle XYZ$ be two triangles which are congruent to each other. Show that there is an isometry f, which is a composition of at most three reflections, such that $f(\triangle ABC) = f(\triangle XYZ)$.
2. Let f be an isometry of the plane, such that f fixes the origin O in a chosen coordinate system. We saw last time that then f can be represented as multiplication of the coordinate vector by a matrix

$$A_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   (a) Show that the entries of the matrix A_f satisfy the following relations:

$$\begin{aligned} a^2 + c^2 &= 1 \\ b^2 + d^2 &= 1 \\ ab + cd &= 0. \end{aligned}$$

   Such a matrix is called *orthogonal*.

   (b) Using the previous part, show that the determinant of A_f must be either 1 or -1.

   (c) Suppose f is a rotation about O. What is the determinant of A_f?

   (d) Suppose f is a reflection about a line containing O. What is the determinant of A_f?

   (e) Given A_f, what are the fixed points of f (other than O)?

3. Let f be any isometry of the plane (it does not have to fix O). We saw last time that f can be represented as multiplication by a matrix

$$A_f = \begin{bmatrix} a & b & v_x \\ c & d & v_y \\ 0 & 0 & 1 \end{bmatrix}.$$

   The determinant of this A_f must also be either 1 or -1. If it is 1, then f is called "orientation preserving," and if it is -1, it is called "orientation reversing."

   (a) Show that any reflection is orientation reversing.

(b) Show that a composition of any two reflections is orientation preserving.

(c) Let $\triangle ABC$ be a triangle. What happens to the ordering of its vertices after applying f, if f is orientation preserving? What if f is orientation reversing?

4. Last week we talked about composing two reflections r_{l_1} and r_{l_2}. Recall what happened:

   (a) If $l_1 \parallel l_2$, then $r_{l_2} \circ r_{l_1}$ is a ________________________.

   (b) If $l_1 \nparallel l_2$, then $r_{l_2} \circ r_{l_1}$ is a ________________________.

5. Consider now the composition of *three* reflections, $f = r_{l_3} \circ r_{l_2} \circ r_{l_1}$.

   (a) If $l_1 \parallel l_2 \parallel l_3$, prove that f is a reflection.

   (b) If $l_1 \parallel l_2$ and $l_3 \perp l_1$, then f is called a *glide reflection.*

   (c) If $l_1 \parallel l_2$ and l_3 intersects l_1 and l_2 at an angle ϕ, what must f be?

   (d) What if another pair of the lines are parallel, and the remaining intersects them at a given angle?

   (e) If l_1, l_2, l_3 intersect at three points A, B, C, then what must f be?

   (f) What if l_1, l_2, l_3 intersect at a common point O?