MATH 402 Non-Euclidean Geometry

Worksheet on hyperbolic geometry

Monday 09/14.

Recall that the Poincaré disk model for the hyperbolic plane is defined as follows:

- *(Hyperbolic) points* are the points in a Euclidean unit disk $\mathcal{D}$ (but not on the boundary circle $\mathcal{C}$), and
- *(Hyperbolic) Lines* are either the (Euclidean) diameters of $\mathcal{D}$, or (Euclidean) circles which are orthogonal to $\mathcal{C}$.

Work in groups on the following questions.

1. Show that through given two hyperbolic points P, Q, there is a unique hyperbolic line. To do this, you need to consider two cases. Let O be the center of $\mathcal{C}$
   (a) If P, Q, O are colinear, then...
   (b) If P, Q, O are not colinear, you need to show that...
2. How must "between-ness" be defined in order for Hilbert's between-ness axioms to be satisfied?
3. Euclid second postulate said that lines can always be extended. Is this true in the hyperbolic disk? Why?
4. Distance in the Poincaré hyperbolic disk is defined as follows. If we want to compute $d_p(P, Q)$, let $\mathcal{S}$ be the hyperbolic line containing P, Q. We need to find the points R, S in the intersection of $\mathcal{S}$ and $\mathcal{C}$. Then we define

$$d_p(P,Q) = \left| \ln \frac{(\overline{PS})(\overline{QR})}{(\overline{PR})(\overline{QS})} \right|.$$

   (a) Show that $d_p(P, Q) = 0$ if and only if $P = Q$.
   (b) What is the distance between a point P and the "origin" O?
   (c) How does $d_p(P, Q)$ change as we keep P fixed and move Q (in a hyperbolic line) towards $\mathcal{C}$?
5. (a) Define a hyperbolic circle.
   (b) Do hyperbolic circles exist? What is their Euclidean shape?
6. How do parallel lines look like in the Poincaré disk?

Remark: Hyperbolic angle measurements are defined as their Euclidean analogues.