# MATH 402 Non-Euclidean Geometry

## Exam 3 Practice Questions

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Other great source for practice questions: the book. More precisely, any exercise or proof from Chapter 7 (7.4, 7.5, 7.6) and 8 (8.1, 8.2); background material on complex numbers and function is in 3.5. In addition, review all homework; solutions are posted on the webpage.

**8.6.1 will not be covered!**

**General hint: make sketches whenever you're asked to do something visual.**

1. Definitions and statements.
   (a) Defect of a hyperbolic triangle.
   (b) The axioms for area.
   (c) Saccheri and Lambert quadrilaterals.
   (d) Scissor congruence (called equivalence in the book).
   (e) Riemann sphere and extended complex plane.
   (f) Stereographic projection.
   (g) Möbius transformation of the extended complex plane.
   (h) Cross ratio of four complex numbers.
2. Proofs and other longer answers
   (a) Show that any hyperbolic triangle is scissor congruent to a Saccheri quadrilateral.
   (b) State and prove the Angle-Angle-Angle congruence in hyperbolic geometry.
   (c) Write formulas for the following type of Möbius transformations:
      i. $f$ which maps the unit circle $|z| = 1$ to the horizontal line $\operatorname{Im} z = 1$.

ii. $f$ which maps the unit circle $|z| = 1$ to the circle centered at $1 + i$ with radius 2.

iii. $f$ which maps the points $1, i$ to the points $-i, -1$.

In each case, how many such Möbius transformations exist?

(d) For the following functions on the extended complex plane, discuss the geometric effect they have.

i. $f(z) = \alpha z$, where $\alpha$ is a fixed complex number

ii. $f(z) = \frac{1}{z}$

iii. $f(z) = \overline{z}$

iv. $f(z) = \frac{1}{\overline{z}-1} + 1$

(e) Prove that the inverse of a Möbius transformation is also a Möbius transformation.

(f) Prove that if $f, g$ are Möbius transformations which agree on three different points $z_0, z_1, z_2$, then $f = g$.

*Hint: first prove that $g^{-1} \circ f$ is a Möbius transformation which fixes the given three points. Then prove that a non-identity Möbius transformation can have at most 2 fixed points.*

(g) Suppose $w_0, w_1, w_2, w_3$ are four points on a line in the complex plane. Show that if $w_1$ is between $w_0$ and $w_2$, and $w_2$ is between $w_1$ and $w_3$, then the cross ratio $(w_0, w_1, w_2, w_3)$ is a *positive* real number. Do that in the following steps:

i. First, prove that if $f$ is a Möbius transformation, then it preserves cross ratios, i.e. for any $z_0, z_1, z_2, z_3$,

$$(f(z_0), f(z_1), f(z_2), f(z_3)) = (z_0, z_1, z_2, z_3).$$

ii. Next, use a Möbius transformation (which one?) to move the points $w_0, w_1, w_2, w_3$ to the real line.

iii. What is the order (i.e. betweenness) among the images

$$f(w_0), f(w_1), f(w_2), f(w_3)?$$

Why?

iv. Conclude therefore what is the sign of the cross ratio

$$(f(w_0), f(w_1), f(w_2), f(w_3)).$$

(h) Let $z$ be a point in the Poincaré disk. Prove that the hyperbolic distance between 0 and $z$ is

$$d_p(0, z) = \ln\left(\frac{1 + |z|}{1 - |z|}\right).$$

3. More practice with complex numbers

   (a) Write the equation of a line through two given points $z_1$ and $z_2$ (without using the real and imaginary parts of $z_1$ and $z_2$ separately.)

   (b) Write the equation of a circle through $z_0$ centered at $a$.

   (c) What is $z\overline{z}$? What is $z + \overline{z}$?

   (d) How can you tell if a complex number $z$ is real? (without using the real and imaginary parts of $z$ separately)

4. True or false, and other brief answer questions.

   (a) There exists a hyperbolic triangle whose defect is exactly 180°.

   (b) What is the defect of an omega triangle? What about a "double omega" triangle? And what about a "triple omega" triangle?

   (c) In the complex plane, how can you write a rotation of $\theta$ about the origin? Use polar form of complex numbers.

   (d) Any Möbius transformation maps lines to lines.

   (e) How many clines go through two given distinct points? What if one of the points is $\infty$?

   (f) Suppose $z_0, z_1, z_2, z_3$ are four distinct points. How can you tell if they are all collinear? How can you tell if they lie on the same circle?

   (g) Suppose $z_0, z_1, z_2, z_3$ are complex numbers corresponding to the vertices of a rectangle. Is the cross ratio $(z_0, z_1, z_2, z_3)$ necessarily real?

(h) Suppose $z_1, z_2, z_3$ are complex numbers corresponding to the vertices of a triangle, and let $z_0 = \frac{z_1+z_2+z_3}{3}$ be the "center of mass" of the triangle. Can the cross ratio $(z_0, z_1, z_2, z_3)$ be real?

5. Solve all exercises from Chapters 8.2 and 3.5.