MATH 402 Non-Euclidean Geometry

Exam 2 Practice Questions

Other great source for practice questions: the book. More precisely, any exercise or proof from Chapter 5 and sections 7.1, 7.2, 7.3 (excluding 5.5 and 7.2.2). In addition, review all homework; solutions are posted on the webpage.

General hint: make sketches for (mostly) everything.

1. Definitions and statements

   (a) Define an isometry of the Euclidean plane.

   (b) Remember all the statements of Theorem 5.1. Write them down.

   (c) Let f be an isometry. Define a fixed point of f.

   (d) Define a reflection about a line m, using fixed points.

   (e) Define a translation using reflections.

   (f) Given the above definition, there is a theorem that tells us how to associate a vector of translation to a translation. State this theorem.

   (g) Define a rotation using reflections.

   (h) Given the above definition, there is a theorem that tells us how to associate an angle of rotation to a rotation. State this theorem.

   (i) Define points and lines in the Poincaré disk model of hyperbolic geometry (aka Poincaré plane).

   (j) Define omega points in the Poincaré plane.

   (k) Define omega triangles in the Poincaré plane.

   (l) What is the distance between two points P, Q in the Poincaré plane?

   (m) Define limiting parallels in the hyperbolic plane.

   (n) State the fundamental theorem of parallels in hyperbolic geometry.

2. Constructions

   (a) Given a hyperbolic line m and a point P, how can one construct a hyperbolic line l through P perpendicular to m, in the following cases:

      i. P is not on m

      ii. P is on m

   (b) Given a hyperbolic line l in the Poincaré plane, and a point P not on l, construct the limiting parallels of l through P.

3. Proofs and other longer answers

   (a) Let l_1, l_2, l_3 be three different lines, and let $f = r_{l_3} \circ r_{l_2} \circ r_{l_1}$ be the composition of the associated reflections.

      i. If $l_1 \parallel l_2 \parallel l_3$, what type of isometry is f? What is the set of fixed points of f? Prove your answer.

      ii. If l_1, l_2, l_3 all intersect at a common point O, what type of isometry is f? What is the set of fixed points of f? Prove your answer.

   Hint: In both of these situations, first compose two of the given reflections, and describe the resulting isometry. Then work on composing that with the remaining reflection.

   (b) The vector $(1, 2)$ defines a translation T of the Euclidean plane in coordinates.

      i. If (x, y) are the coordinates of any point, what are the coordinates of $T(x, y)$?

      ii. A translation is a composition of two reflections, so $T = r_{l_2} \circ r_{l_1}$ for some lines l_1, l_2. Describe in words what are the lines l_1 and l_2 associated to our translation T?

      iii. Write down the equations in coordinates of the lines l_1, l_2.

   (c) Suppose f is a transformation of the plane, which takes the following values on a few chosen points.

(x,y)	(0,0)	(0,1)	(1,1)	(2,1)
f(x,y)	(1,1)	(2,2)	(2,3)	(-1,0)

   Can f be an isometry? Prove your answer.

(d) Let $\mathcal{R}$ be the set of all rotations of the plane around a fixed point O. Show that $\mathcal{R}$ forms a group.

(e) Let f be an isometry of the Poincaré disk, which preserves two different omega points Ω_1 and Ω_2. What must f be? Prove your answer.

Hint: use the property of omega points given in Definition 7.7 of the book.

(f) Suppose l is a hyperbolic line, and P a point not on l. Let m be a limiting parallel to l through P. Let Q be any other point on m. Show that m is a limiting parallel to l also through Q.

4. True or false?

(a) If l_1 and l_2 are two lines, the composition of reflections $r_{l_2} \circ r_{l_1}$ could be a reflection, depending on the mutual position of l_1 and l_2.

(b) If l_1 and l_2 are two lines, the composition of reflections $r_{l_2} \circ r_{l_1}$ could be a rotation, depending on the mutual position of l_1 and l_2.

(c) If l_1 and l_2 are two lines, the composition of reflections $r_{l_2} \circ r_{l_1}$ could be a translation, depending on the mutual position of l_1 and l_2.

(d) Let f be any isometry, and let r_m be a rotation about a line m. It is possible that $f \circ r_m \circ f^{-1}$ is a reflection.

(e) Are the following functions on the Euclidean plane, given in coordinates, isometries?

i. $f(x, y) = (x, -y)$

ii. $f(x, y) = (x - 1, -y + 1)$

iii. $f(x, y) = (-x, xy)$

iv. $f(x, y) = (e^x, -y)$

(f) In the Poincaré disk, given a line l and point P not on l, there are exactly two lines through P which are parallel to l.

(g) Hyperbolic reflections preserve hyperbolic distances.

(h) Hyperbolic reflections preserve hyperbolic angles.

(i) The relation “is parallel to” is transitive in hyperbolic geometry.