The shallow water equations are made rigorous under the assumption that \((\frac{h_0}{\lambda^2}) \ll 1\). This indicates that the ratio of the depth from the bottom 'ho' to the length of the waves '\lambda' is considerably small, and these are therefore also known as long waves.

\[
\begin{cases}
\frac{\partial h}{\partial t} + (U h)_x = 0 \\
u_t + g h_x + U U_x = 0
\end{cases}
\]

These can be solved using the method of Riemann invariants, as given in Chapter 5 of [Whit].

\[
\begin{align*}
h(x, t) &= f(s) \\
u(x, t) &= 2 \sqrt{g h_0} - 2 \sqrt{g h_0} \\
x(s, t) &= s + \left[ 3 \sqrt{g h_0} - 2 \sqrt{g h_0} \right] t
\end{align*}
\]

There is a shock at \(T_{br} = \inf_{s \in \mathbb{R}} \left[ \frac{2}{3} \frac{1}{\sqrt{g f(s)}} \right] \).

\[
\therefore \text{ whenever } f(s) < 0, \text{ there is a shock, which leads to breaking. This indicates that all non-trivial solutions break, which is an undesirable property for a realistic model of water waves since long waves which do not break have been observed in nature, & in experiments.}
\]

One explanation for this is that as time \(t\) approaches the breaking time \(T_{br}\), the slope of waves become steeper, violating the shallow water assumption of \(\frac{h_0^2}{\lambda^2} \ll 1\).
This implies that the solution isn't valid in this regime. However, this method still gives good approximate results in many specific cases, one example being the dam breaking problem. (Chapter 13 [Whi.3])

A "good" theory should give examples where certain waves break while others don't. To overcome the problem of all non-trivial waves breaking, we introduce dispersion into our model.

Boussinesq theory

Here we go back to the linearized dynamics obtained from the linearized system.

\[ C^2(k) = g \tanh(kh_0) \frac{k}{k} \]

Hence \( C^2(k) = \frac{g}{k} \) (Chapter 13). Hence \( C^2(k) \ll C_0^2 \)

In shallow water \( \frac{h_0}{\ell} \ll 1 \), \( \ell \ll \frac{1}{k} \). Hence \( (kh_0)^2 \ll 1 \)

\[ C^2(k) \approx \frac{g}{k} (1 - \frac{1}{3} k^2 h_0^2 + \ldots) \] by expanding using Taylor series.

Collecting the \( O(1) \) terms,

\[ C^2(k) = gh_0 = C_0^2 \]

The simplest model with this dispersion relation is the wave equation.

\[ \nabla^2 \eta - C_0^2 \nabla \eta_{,tt} = 0 \]  \( \text{(3)} \)

Functions of the type \( \eta(x,t) = A \cos(k(x-ct)+\phi) \)

with \( C^2=C_0^2 \), \( A, \phi \) constants are solutions of this.

This can also be obtained as the linear part of the shallow water equations.
let the total height \( h = \eta + h_0 \)

then the shallow water equations \( \text{O}(1) \) give

\[
\begin{align*}
\eta_t & = -g \eta_x - u \eta_x = -g \eta_x - U \eta_x \\
U_t & = -gh_x - u U_x = -g \eta_x - U \eta_x
\end{align*}
\]

considering \( \text{O}(3) \) equations,

\[
\begin{align*}
\eta_{tt} & = -\left( 3U \left( 3\eta + h_0 \right) \right)_x & \text{and} \quad U_t &= -g(3\eta)_x - 3 h \eta_x - 3 \eta h \eta_x
\end{align*}
\]

lead to

\[
\eta_t + h_0 U_x = 0 \quad \text{and} \quad U_t + g \eta_x = 0
\]

by collecting \( \text{O}(3) \) terms.

but this system gives us

\[
\begin{align*}
\eta_{tt} &= -h_0 U_{xt} & \text{and} \quad U_{xx} &= -g \eta_{xx}
\end{align*}
\]

by taking partial derivatives. Since partials commute under sufficient regularity, \( U_{xt} = U_{tx} \)

\[
\frac{\eta_{tt}}{-h_0} = -g \eta_{xx} \quad \text{from which we can recover the wave equation} \ 3.
\]

Taking the \( \text{O}(k^2) \) terms, also into account,

\[
C^2(k) = C_0^2 \left( 1 - \frac{1}{3} k^2 h_0^2 \right), \quad \text{which is obtained from a Fourier multiplier of the type} \ 1 - \frac{1}{3} h_0^2 \partial^{xxx} \partial.
\]

This leads us to the equation

\[
\eta_{tt} - C_0^2 \eta_{xx} - \frac{1}{3} C_0^2 h_0^2 \eta_{xxxx} = 0, \quad \text{---(4)}
\]
This is equivalent to the system,
\[ \begin{cases} n_t + n_x = 0 \\ n_t + n_x + \frac{1}{3} \frac{\partial}{\partial x} n^2 = 0 \end{cases} \]

The equivalence can be seen by taking suitable partial derivatives of the latter & using the fact that partials commute \([n x = \frac{\partial}{\partial x} n] \), similar to the equivalent system of the wave equation.

This leads us to the “bad Boussinesq” system.

\[ \begin{cases} h_t + (h u)_x = 0 \\ u_t + 2 g h + \frac{1}{3} C^2 h h x x x + 2 U U_x = 0 \end{cases} \]

obtained by adding non-linear terms to make it similar to the shallow water equations.

There is no proof for the well-posedness of this system, but this is still used in certain applications. The higher order term is hard to control.

Using \( C^2 \approx \frac{2}{3} \), this can be transformed into what we shall call the “Good Boussinesq” system.

\[ \begin{cases} h_t + (h u)_x = 0 \\ u_t + g h + \frac{1}{3} C h h x x + U U_x = 0 \end{cases} \]

By changing the higher order term, we get this formally equivalent system which is well posed.

Using \( U \approx C n \), we can even turn the latter equation to

\[ u_t + g h + \frac{1}{3} n h U x + U U_x = 0. \]

This has much better properties since \([1 + \frac{1}{3} h^2] \) gives smoothing.
These models are bidirectional and give us waves propagating in both directions. By taking one of these parts, we recover hyperbolic unidirectional waves.

Korteweg–de Vries theory

\[ C(k) = \frac{1}{k} \sqrt{g \tanh(kh_0)} \]

is the dispersion relation from the linearized system that is chosen.

When \( kh_0 \ll 1 \), as before

\[ C(k) \approx C_0 \left( 1 + \frac{1}{6} k^2 h_0^2 \right) \quad C_0 = \sqrt{gh_0} \]

Once again, the simplest equation such that this is a solution is given by

\[ \eta_t + C_0 \eta_x + \frac{1}{6} C_0 h_0^2 \eta_{xxx} = 0 \quad \text{---8} \]

Once again, by making it non-linear to match the shallow water equations, and converting the variable \( \eta \) to \( \eta_0 \) using \( \text{1} \& \text{2} \),

\[ \eta_t + (\eta + h_0 \eta_t) \eta_x = 0 \]

where \( \eta = 2 \sqrt{g(h_0 + \eta)} - 2 \sqrt{gh_0} \) from the solution to the shallow water equations, we get which gives us, by substitution & taking the derivative,

\[ \eta_t + (3 \sqrt{g(h_0 + \eta)} - 2 \sqrt{gh_0}) \eta_x = 0 \]

we get

\[ \eta_t + (3 \sqrt{gh_0 + \eta_0}) - 2 \sqrt{gh_0}) \eta_x + \frac{1}{6} C_0 h_0^2 \eta_{xxx} = 0 \quad \text{---9} \]

the 2nd term can be reduced to \( 3 C_0 \left[ \sqrt{1 + \frac{1}{6} \eta_0} + 0 \left( \frac{\eta_0^2}{gh_0} \right) \right] \eta_x \)

if \( \frac{\eta_0}{h_0} \ll 1 \) where \( a \) is the typical amplitude. This allows us to assume \( \frac{\eta_0}{h_0} \ll 1 \).
This gives us the KdV equation:

\[ u_t + 4u u_x + \frac{1}{6} \left( \frac{1}{1} \right) u_{xxx} + \frac{3}{2} \sqrt{u_{xx}} u_{xx} = 0 \quad - (10) \]

The two main assumptions used here are the shallow water & small amplitude assumptions.

For rigorous justification of a model, we need to compare the solutions with the actual water wave system of equations. This is done by first establishing the local wellposedness of the water wave system up to some time \( T \), in a suitable space of functions. Then, wellposedness of the model can be established for the same time, & the solutions can be compared in the same space.

For the KdV system, we can justify the model when

\[ \left( \frac{h_0^2}{h^2} \right) = \frac{a}{h_0} \]

The Boussinesq system \( \Box \) has dispersive relation

\[ C^2(k) = \frac{C_0}{1 - \frac{1}{6} k^4 h_0^2} \]

which has the same Taylor expansion up to order \( k^2 \) as

Experiments in actual water indicate that \( C^2(k) \propto \frac{1}{k} \) is a more appropriate dispersive relation to have since \( C \) decays as the wave number \( k \) grows, which indicates \( \Box \) is closer than \( \Box \) to the actual case.

Hence, if we try \( C(k) = \frac{C_0}{1 - \frac{1}{6} k^4 h_0^2} \) at the start, we get

\[ u_t + 4u u_x - \frac{1}{6} \left( \frac{1}{1} \right) u_{xxx} + \frac{3}{2} \sqrt{u_{xx}} u_{xx} = 0 \quad - (11) \]

This is the Benjamin-Bona-Mahoney (BBM) equation.
Numerical solutions show that this is better than the Boussinesq equations.

The KdV equation has the dispersion relation $\sigma(1 + \frac{1}{2} k^2 h_0^2)$

Since $2 \alpha \frac{1}{k}, (1 + \frac{1}{2} k^2)$ grows rapidly as $\alpha$ becomes smaller. Hence short waves are dispersed fast. The KdV equation admits soliton solutions but none of the solutions break. This is again a bad feature.
Looking for the simplest model which admits both breaking and non-breaking solutions, one finds the Whitham equation.

**Whitham equation.**

Here, the idea is to replace $\left[ \sigma(1 + \frac{1}{2} k^2 h_0^2) \right]$ terms in the KdV [A BBM] equation to get the waterwave dispersion relation. This is done by introducing the pseudo-differential operator $M$.

$$M \psi(x) := C_{ww}(k) \psi(x)$$

where $C_{ww}(k) = \frac{g \tanh(kh_0)}{k}$

This gives us the Whitham equation,

$$\frac{\partial}{\partial t} + M \psi + \frac{3}{2} \sqrt{\frac{g}{h_0}} \frac{\partial}{\partial x} \psi_\psi \frac{\partial}{\partial x} = 0 \quad (12)$$

$$M \psi(x) = \int_{-\infty}^{\infty} K_{ww}(x-y) \psi(y) \frac{\partial}{\partial x} \psi \frac{\partial}{\partial x} dy$$

gives the dispersion,

where $K_{ww}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{ww}(k) e^{ikx} dk$

A few properties of this kernel can be found in section 13.19 [Whitham]
A natural generalization of Whitham's equation is obtained by replacing the integral operator $K_{uv}$ with an arbitrary pseudo-differential operator $K$. In 1968, Seliger was able to prove that a sufficiently asymmetric initial hump would break for certain $K$. The most important assumption here was that $|K(t)| < +\infty$. He couldn't generalize this for $K_{uv}$.

Indeed, the following conditions are necessary.

1. $K(t)$ is even symmetric.
2. $K \geq 0$ as $|x| \to \infty$.
3. $|K(t)| < +\infty$.

The main idea of the proof is to get a differential inequality, giving a breaking time, similar to the shallow water equations. We provide a sketch of the proof.

Let $m_{+}(t) := \max_{x} \eta_{x} \text{ at } (x = x_{+}(t)) = \eta_{x}(x_{+}(t), t)$

$m_{-}(t) := \min_{x} \eta_{x} \text{ at } (x = x_{-}(t)) = \eta_{x}(x_{-}(t), t)$

Then, by differentiating the equation we obtain

$$
\left( \frac{dm_{\pm}}{dt} \right) + \frac{3}{2} m_{\pm}^{2} + \int_{-\infty}^{\infty} K(y) \eta_{yy}(x_{\pm}(t) - y, t) \, dy = 0
$$

The integrals can be estimated using the appropriate mean value theorem, found in

$$
\int_{-\infty}^{\infty} K(y) \eta_{yy}(x_{\pm} - y, t) \, dy \leq |K(t)| (m_{+}(t) - m_{-}(t))
$$

This gives us two differential inequalities, which when summed up leads to

$$
\frac{d}{dt} (m_{+} + m_{-}) \leq -\frac{3}{2} (m_{+}^{2} - m_{-}^{2}) + 2 |K(t)| (m_{+} - m_{-})
$$
This simplifies to
\[
\frac{d}{dt} (m_+ + m_-) \leq (m_+ - m_-) \left[ 2K(0) + \frac{3}{2} (m_+ + m_-) \right] - 3m_+^2
\]

If we now consider an initially asymmetric wave, which satisfies
\[
[m_+ (0) + m_- (0)] \leq -\frac{4}{3} K(0), \quad \text{the inequality tells us that}
\]
\[
[m_+ (t) + m_- (t)] \leq -\frac{4}{3} K(0) \quad t \geq 0.
\]

We can use this latter estimate to replace \( m_+ \) in our differential inequality for \( m_- \)
\[
\frac{d}{dt} m_- \leq -\frac{3}{2} m_-^2 + K(0) [m_+ - m_-]
\]
\[
\leq -\frac{3}{2} m_-^2 - 2K(0) m_- - \frac{4}{3} K(0)
\]
\[
= -\frac{3}{2} \left[ m_- + \frac{2}{3} K(0) \right]^2 - \frac{2}{3} K(0)
\]
\[
\frac{d}{dt} \left[ m_- + \frac{2}{3} K(0) \right] \left( \frac{3}{2} \right) + \left( \frac{3}{2} \right) \left[ m_- + \frac{2}{3} K(0) \right]^2 \leq -\frac{2}{3} K(0) \leq 0
\]

which shows us that \( \left( \frac{3}{2} m_- + \frac{2}{3} K(0) \right) \) approaches \( \infty \) in finite time, less than
\[
\bar{T} = \frac{1}{\frac{3}{2} m_- (0) + K(0)}
\]

That is, the asymmetry increases with time until breaking occurs.

Details can be found in chapter 13 [Whi85], [SEL], [C-E].

Whitham conjectured that the equation he derived will explain peaking & breaking. Results have been found for this, even after Sohier's proof. Some other results are given in [C-E], [EW16] where peaking was demonstrated & in [HJK17] where breaking was explained.
References


[EW] M. Ehrnström & E. Wahlén, On Whitham’s conjecture of a highest cusped wave for a non-local dispersive equation, arXiv:1602.05389v1