The vector field \( \vec{u} := (u, v) \) is the velocity vector field. For \(-h_0 < y < \eta(x, y)\) the components of \( \vec{u} \) satisfy the system of PDEs:

\[
\begin{aligned}
    u_t + uu_x + vu_y &= -p_x \\
    u_t + uv_x + v_y &= -p_y - y \\
    u_x + v_y &= 0.
\end{aligned}
\]

(1)

The surface of the liquid is governed by the kinematic boundary condition \( \eta_t + u\eta_x = v \) and the dynamic boundary condition \( p = p_{atm} \) both on \( y = \eta(x, t) \). We also impose that \( v = 0 \) on \( y = -h_0 \).

The curl of the velocity vector field is the vorticity \( \omega \). In 2-dimensions, the vorticity is \( \omega = v_x - u_y \) and satisfies

\[
\omega_t - (\vec{u} \cdot \nabla)\omega = 0.
\]

This implies that the vorticity is only transported! Thus, the total vorticity is also conserved and if \( \omega(0) \) is constant then it is constant for all time. This is known as Kelvin’s circulation theorem. Note that the last equation in (1) together with the zero vorticity condition form the Cauchy-Riemann equations.

It follows that there is a \( \phi \) such that \( \nabla \phi = (u, v) \). The function \( \phi \) is called the velocity potential. Again by the Cauchy-Riemann equations \( \Delta \phi = 0 \) in \(-h_0 < y < \eta(x, t)\). The velocity potential then satisfies the system

\[
\begin{aligned}
    \phi_t + \frac{1}{2} |\nabla \phi|^2 + gy + p &= b(t) \quad \text{on} \quad y = (x, t) \quad (\text{Bernoulli’s Eqn.}) \\
    \eta_t + \phi_x \eta_x = \phi_y \phi_y &= 0 \quad \text{on} \quad y = -h_0.
\end{aligned}
\]

Linear Theory:

Now we study the system (1) by linearizing about the trivial solution \( \eta(x, t) \equiv 0 \), \( (u, v) \equiv 0 \), \( p = p_{atm} - gy \). The perturbations are then of the form

\[
\eta = 0 + \epsilon \tilde{\eta}, \ u = 0 + \epsilon \tilde{u}, \ v = 0 + \epsilon \tilde{v}, \ p = p_{atm} - gy + \epsilon \tilde{p}.
\]

Expanding the solution in powers of \( \epsilon \) we find that the \( \epsilon^0 \) (constant) term is the trivial solution, the \( \epsilon^1 \) coefficient is the main contribution to the perturbation, and the \( \epsilon^2 \) terms
and higher are small contributions. The main contribution solves the following system for \(-h_0 < y < 0:\)

\[
\begin{align*}
  u_t &= -p_x \\
  v_t &= -p_y \\
  u_x + v_y &= 0 \quad \text{and} \quad v_x - u_y &= 0
\end{align*}
\]

and on the boundary the solution satisfies

\[
\begin{align*}
  v &= \eta_t \quad \text{and} \quad p = g\eta \quad \text{on} \quad y = 0 \\
  v &= 0 \quad \text{on} \quad y = -h_0.
\end{align*}
\]

Next we look for solutions of the form:

\[\eta(x, t) = \cos(k(x - ct)).\]

After using this ansatz we find that

\[
\begin{align*}
  u(x, y, t) &= \frac{ck \cosh(k(y + h_0))}{\sinh(kh_0)} \cos(k(x - ct)) \\
  v(x, y, t) &= \frac{ck \sinh(k(y + h_0))}{\sinh(kh_0)} \sin(k(x - ct)) \\
  p(x, y, t) &= \frac{c^2k \cosh(k(y + h_0))}{\sinh(kh_0)} \cos(k(x - ct)).
\end{align*}
\]

It follows that we must have the dispersion relation \(c^2 = g \tanh(kh_0)\). If instead you use the ansatz \(e^{i(k(x-ct))}\) it can happen that \(c = |c|i\) is purely imaginary. This means the linear part of the equation is behaving like a parabolic equation since \(e^{i(k(x-ct)}) = Ae^{-|c|t}\).

**Surface Tension**

If we impose that there is surface tension at the surface of the water then this means that the pressure no longer satisfies the \(p = p_{atm}\) boundary condition. Instead, the jump in pressure across the interface is is proportional to the curvature so that

\[p = T \cdot \left( \frac{1}{\sqrt{1 + \eta_x^2}} \right)_x,\]

where \(T\) is the *surface tension coefficient*. Note the \(p_{atm}\) does not appear here because of the linearization. After the same ansatz calculation we find that

\[c^2(k) = (g + Tk^2) \frac{\tanh(kh_0)}{k}.\]

In the ocean with \(g \approx 9.8 \text{ m/sec}^2\), it is believed that \(T \approx 7.3 \times 10^{-3} \text{ N/M},\) although there is significant disagreement in the value of this constant.

\[
c^2(k) \approx \begin{cases} 
  g \frac{\tanh(kh_0)}{k} & g >> T \\
  Tk \frac{\tanh(kh_0)}{k} & T >> g.
\end{cases}
\]

The wavelength \(\lambda = \frac{2\pi}{k} \approx 1.7 \text{ cm}.\)
Shallow Water Wave Theory

Assume that \( v \approx 0, v_t \approx 0, v_x \approx 0 \). Then we have that \( p_y \approx -g \) which implies that \( p = p_{atm} - g(\eta - y) \). Then we have that

\[
u_t + uu_x + vu_y = -g\eta_x.
\]

Note that the right-hand-side is independent of \( y \). Also assume that

\[
\frac{Du}{dt} = (\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u})(0)
\]
is also independent of \( y \). This implies that \( \frac{Du}{dt} \) is independent of \( y \) for all \( t \).

\[
u_t + g\eta_x + uu_x = 0
\]

\[
0 = \int_{-h_0}^{\eta(x,t)} u_x + u_y \, dy
\]

\[
= \left( \int_{-h_0}^{\eta(x,t)} u \, dy \right) - u(x, \eta(x,t), t)\eta_x(x,t) + u(x, -h_0, t)(h_0)_x + v(x, \eta(x), t) - v(x, -h_0, t)
\]

\[
= \left( \int_{-h_0}^{\eta(x,t)} u \, dy \right) + \eta_t(x,t)
\]

(2)

By assumption \( u \) is independent of \( y \). This implies (2) is \( (u(\eta + h_0))_x \). Now we have the shallow water wave equations:

\[
\begin{align*}
\eta_t + (u(\eta + h_0))_x &= 0 \\
u_t + g\eta_x + uu_x &= 0 \\
\eta &= 0, u = -\omega y, v = 0 \\
\vec{u} &= (-\omega y + u, v)
\end{align*}
\]

Now set \( h = \eta + h_0 \) so that \( h_t + (uh)_x = 0 \). We will attempt to solve these equations by using a modified method of characteristics. To start recall that Burger’s equation is

\[
u_t + uu_x = 0.
\]

Using the usual method of characteristics, we search for a function \( x(t) \) such that \( \frac{dx}{dt} = u(x, t) \) so that \( \frac{d}{dt}(u(x(t), t)) = 0 \) by using Burger’s equation. This shows that \( u(x(t), t) \) is a constant.

In our case, we can rewrite our shallow water wave equations as the vector equation

\[
\begin{bmatrix}
h \\ u
\end{bmatrix}_t + \begin{bmatrix}
u & h \\ g & u
\end{bmatrix} \begin{bmatrix}
h \\ u
\end{bmatrix}_x = 0.
\]

The eigenvalues of the above matrix are the characteristic velocities which are \( \lambda_\pm = u \pm \sqrt{gh} \). The characteristic curves are the solutions of \( \frac{dx}{dt} = \lambda_\pm(x) \). The left eigenvectors of the matrix are \((g, \pm \sqrt{gh})\). Then the system is equivalent to

\[
gh_t \pm \sqrt{ghu_t} + \lambda \pm (gh_x \pm \sqrt{ghu_x}) = 0.
\]
Now we compute the *Riemann invariants* $r_\pm$, which satisfy $\frac{dr_\pm}{dt} = 0$ along the characteristics. We look for an integrating factor $\mu$ such that $\mu \cdot (\text{char. form}) = 0$. It is possible to compute and find $r_\pm = u \pm 2\sqrt{gh}$.

\[
\begin{cases}
  h(x, t) = f(s) \\
  u(x, t) = 2\sqrt{gh_0} - 2\sqrt{gf(s)} \\
  x(s, t) = s + t(3\sqrt{gf(s)} - 2\sqrt{gh_0}).
\end{cases}
\]

We only use $r_+$ because the wave is moving to the right. Similarly to Burger’s equation there are shocks since $x(s)$ is governed by

\[
\frac{\partial x}{\partial s} = 1 + 3 \left( \frac{g'}{2} \right) \frac{\sqrt{g'(s)}}{t} = 0,
\]

which does not have global in $s$ solutions. Solutions $u$ to this differential equation remain bounded, but develop an unbounded first derivative. We interpret this type of “blow-up” of the first derivative as *wave breaking*.

We define the *breaking time* as

\[
t_{Br} = -\inf \left\{ \frac{2}{3} \frac{\sqrt{g'(s)}}{f'(s)} \right\}.
\]

This quantity is proportional to the reciprocal of the maximum of $f'$. If you do the calculation for constant vorticity you find that $t_{Br}$ is proportional to $\frac{1}{|w|}$ for fixed initial data.

**References**
