1. KdV Soliton

We look for solutions to the Korteweg-de Vries (KdV) initial value problem:

\[ u_t + u_x + u_{xxx} + uu_x = 0, \quad u(x,0) = u_0(x) \]  

in the form of a traveling wave (ie soliton):

\[ u(x,t) = \phi(x - (c + 1)t) = \phi(z), \quad \text{where } u(x,t) \to 0 \text{ as } |x| \to \infty \]

Substituting into (1), integrating (and using the decay assumption) gives the \( \phi \)KdV equation:

\[ \phi'' - c\phi + \frac{1}{2}\phi^2 = 0 \]

This can be solved (assuming \( \phi \geq 0 \) for example, we can multiply by \( \phi' \), integrate, then substitute \( \psi^2 = 3c - \phi \)):

\[ \phi(z) = 3c \sech^2 \left( \frac{1}{2} \sqrt{c}(z - z_0) \right), \quad z_0 \text{ arbitrary} \]

**Question:** Stability? If \( u(x,0) \approx \phi(x) \), is \( u(x,t) \approx \phi(z) \)?

First we need some background on the KdV equation.

2. KdV Properties

(1) Conservation Laws

- Hamiltonian: \( H(u) = \int_{-\infty}^{\infty} (u_x)^2 - \frac{1}{2}u^3 dx \)

\[ \frac{d}{dt}H(u) = 0 \text{ using (1), integration by parts, and decay assumption.} \]

Note: \( |H(u)| \lesssim \|u_x\|^2_2 + \|u\|_{\infty}\|u\|^2_2 \lesssim \|u\|^3_{H_1} \) by Sobolev Embedding

Here \( a \lesssim b \) means \( a \leq Cb \) for some absolute constant \( C > 0 \).

- Momentum: \( P(u) = \int_{-\infty}^{\infty} u^2 dx = \|u\|^2_2 \)
2

STABILITY OF KDV SOLITARY WAVES

-Infinity many more conserved quantities (Krusckal, Miura, Gardner ’68,)

(2) \( \phi \) solves \( \delta H = 0 \) with \( P \) fixed

(3) In fact, more is true. \( \phi \) is not just a critical point for \( H \), it can be shown that \( \phi \) is the global minimized of \( H \) subject to \( P = \text{constant} \).

3. Stability of \( \phi \)

Benjamin [1] adapts Boussinesq’s idea to use \( \Delta H := H(u) - H(\phi) \) as a measure of ‘closeness’ between \( u(x,t) \) and \( \phi(z) \). To this end, the aim is to show

\[
\alpha d^2_I(u, \phi) \geq \Delta H \quad \text{at } t = 0
\]

\[
\beta d^2_{II}(u, \phi) \leq \Delta H \quad \text{for } t \geq 0
\]

for some appropriate choice of metrics \( d_I \) and \( d_{II} \) (\( \alpha \) and \( \beta \) are positive constants, possibly involving \( c \)).

The \( H^1 \) Sobolev norm may seem a natural candidate for \( d_I \) and \( d_{II} \), but there is a problem - the perturbed solution may no longer travel at speed \( c \), so we cannot expect \( \|u(x,t) - \phi(x - (c + 1)t)\|_{H^1} \) to remain small.

We avoid this problem by requiring only that \( u \) remain close to \( \phi \) in shape, so define

\[
d(f, g) = \inf_{x_0 \in \mathbb{R}} \|f(x - x_0) - g(x)\|_{H^1}
\]

and use \( d_I = \| \cdot \|_{H^1} \), \( d_{II} = d \). We’ll sometimes write \( d(v) \) to mean \( d(v, 0) \).

\textbf{Proof:} We assume that the perturbed solution \( u(x,t) \) still satisfies \( P(u) = P(\phi) \). Set \( v = u - \phi \). Then at \( t = 0 \):

\[
0 = P(u) - P(\phi) = \int 2\phi v + v^2 dx
\]
\[
\Delta H = H(u) - H(\phi) = \int (2\phi_x v_x - \phi^2 v) + (v_x^2 - \phi^2 v) - \frac{1}{3} v^3 \, dx
\]
\[
= \int -2\phi'' v - \phi^2 v + (v_x^2 - \phi^2 v) - \frac{1}{3} v^3 \, dx
\]
\[
= \int (c - \phi) v^2 + v_x^2 - \frac{1}{3} v^3 \, dx
\]
\[
\lesssim \max(1, c) \|v\|_{H^1}^2 + \frac{1}{3} \|v\|_{H^1}^3
\]

where we used integration by parts, the \( \phi \)KdV equation, (5), \( \phi \geq 0 \), and Sobolev Embedding. Since we are taking \( \|v\| \) small to prove stability, this establishes (3).

The lower bound is more difficult, so we only offer a sketch here. First separate into \( v = v_{even} + v_{odd} \), which we abbreviate as \( v = v_e + v_o \). Then:
\[
\Delta H = 2 \int_0^\infty [(v_e)_x]^2 + (c - \phi)v_e^2 \, dx + 2 \int_0^\infty [(v_o)_x]^2 + (c - \phi)v_o^2 \, dx - \frac{1}{3} \int_{-\infty}^\infty v^3 \, dx
\]
\[
:= I + II - \frac{1}{3} \int_{-\infty}^\infty v^3 \, dx
\]

For the even term \( I \), we can rewrite (5) to see
\[
\int_0^\infty \phi v_e \, dx = -\frac{1}{2} \int_{-\infty}^\infty v^2 \, dx.
\]

Now consider the linear operator
\[
L f = -f_{xx} + (c - \phi) f
\]
and the bilinear form
\[
B(f, g) = \langle L f, g \rangle_{L^2(0,\infty)} = \int_0^\infty f_x g_x + (c - \phi) f g \, dx.
\]

We examine the auxiliary eigenvalue problem
\[
\theta'' + (20 \operatorname{sech}^2(\frac{1}{2}\sqrt{c} z) + \lambda) \theta = 0 \quad \text{on} \quad [0, \infty)
\]
\[
\theta'(0) = 0, \quad \theta \text{ even}
\]
\[
(6)
\]

The spectrum for (6) is known (cf. Titchmarsh, ’62). There are two negative eigenvalues at -16 and -4, and a continuous spectrum from 0 to \( \infty \). Let \( \theta_1 \) and \( \theta_2 \) be eigenfunctions corresponding to the two negative eigenvalues (both involve \text{sech} and \( \phi \) can be written as a linear combination of \( \theta_1 \) and \( \theta_2 \)).
Then we can decompose $f \in L^2[0, \infty)$ as

$$f = f_1 \theta_1 + f_2 \theta_2 + \tilde{\theta}$$

By keeping track of these functions carefully, Benjamin is able to make the comparison with $I$ and establish the lower bound

$$I \geq \frac{1}{4} \int_{-\infty}^{\infty} [(v_e)']^2 + cv_e^2 dx - \frac{2}{5} c^4 \|v_e\|^3_{H^1}$$  \hspace{1cm} (7)

For the odd term $II$, we choose $x_0(t)$ so that

$$\inf_{y \in \mathbb{R}} \int_{-\infty}^{\infty} [u(x,t) - \phi(x-y)]^2 dx = \int_{-\infty}^{\infty} [u(x,t) - \phi(x-x_0(t))]^2 dx$$

It seems Benjamin gave insufficient justification that this infimum is always attained (so that $x_0(t)$ is well-defined). Bona established this in [2].

With this choice of translated $\phi$ [i.e. $v(x,t) := u(x,t) - \phi(x-x_0(t))$], we have

$$d(u, \phi) = \inf_{y \in \mathbb{R}} \|u(x,t) - \phi(x-y)\|_{H^1} \leq \|u(x,t) - \phi(x-x_0(t))\|_{H^1} = \|v\|_{H^1}$$

Now we differentiate the left integral above with respect to $y$ and evaluate at $y = x_0$:

$$0 = \int_{-\infty}^{\infty} 2(u(x,t) - \phi(x-x_0))\phi'(x-x_0) dx$$

$$= \int_{-\infty}^{\infty} 2(v_e + v_o)\phi' dx$$

$$= \int_{-\infty}^{\infty} 2v_o\phi' dx \hspace{1cm} \text{as } \phi' \text{ is odd}$$

which shows that $v_o \perp_{L^2} \phi'$.

This time we consider the eigenvalue problem:

$$\theta'' + (\phi + \lambda c)\theta = 0 \hspace{1cm} \text{on } [0, \infty)$$

$$\theta(0) = 0, \hspace{1cm} \theta \text{ odd}$$  \hspace{1cm} (8)

which is known to have only one negative eigenvalue at -1, with corresponding eigenfunction equal to a constant times $\phi'$, and a continuous spectrum of positive eigenvalues. Thus, for any $f \in L^2$ vanishing at the origin, we can decompose as

$$f = f_1 \theta' + \tilde{f} \hspace{1cm} \text{where } \int (\tilde{f}_x)^2 - \phi \tilde{f}^2 dx \geq 0.$$  

Taking $f = v_o$ and using $v_o \perp_{L^2} \phi'$ to see that $f_1 = 0$ gives immediately

$$\int_{-\infty}^{\infty} [(v_o)_x]^2 - \phi v_o^2 dx \geq 0.$$  \hspace{1cm} (9)
Then
\[ II = \int_{0}^{\infty} \left[ 2[(v_{0})_{x}]^2 + 2(c - \phi)v_{0}^2 \right] dx \]
\[ = \int_{0}^{\infty} \frac{1}{2}(3c - \phi)v_{0}^2 + \frac{1}{2}((v_{0})_{x})^2 + cv_{0}^2 + \frac{3}{2}((v_{0})_{x})^2 - \phi v_{0}^2 \] \[ dx \]
Since \( \phi \leq 3c \), \( c \geq 0 \) and using (9), we see that
\[ II \geq \frac{1}{4} \int_{-\infty}^{\infty} [(v_{0})_{x}]^2 + cv_{0}^2 dx. \]
Now since
\[ \|v\|_{H^1}^2 = \int_{-\infty}^{\infty} [(v_{e})_{x}]^2 + v_{e}^2 + [(v_{0})_{x}]^2 + v_{0}^2 dx = \|v_{e}\|_{H^1}^2 + \|v_{0}\|_{H^1}^2, \]
and by using Sobolev embedding for the cubic term, we can find \( \beta_1, \beta_2 > 0 \) (depending on \( c \)) so that:
\[ \Delta H \geq \beta_1 \|v\|_{H^1}^2 - \beta_2 \|v\|_{H^1}^3 \]
The function \( \beta_1 x^2 - \beta_2 x^3, x \geq 0 \) increases up to a maximum of \( \frac{4}{27} \frac{\beta_1^3}{\beta_2^2} \) at \( x = \frac{2\beta_1}{3\beta_2} \), then decreases. By the upper bound (3), \( \Delta H \) can be made smaller than this maximum. Then, assuming \( \|v\|_{H^1} \) varies continuously with time, this forces \( \|v\|_{H^1} \leq \frac{2\beta_1}{3\beta_2} \).
Finally, recall that our choice of \( v \) ensures \( d(u, \phi) \leq \|v\|_{H^1} \), so we have established stability.

The continuous dependence of \( \|v\| \) on time was also established by Bona in [2].

References