1 (Holmes; 2013, 3.2(a)). Find a first-term expansion of the solution to
\[ y'' + \sin(y) = 0, \quad y(0) = \epsilon, \quad y'(0) = 0, \]
which is valid for large \( t \).

2 (Holmes; 2013, 3.17). Consider \( y'' + y + \epsilon y^3 = 0 \) subject to the initial conditions \( y(0) = a, \ y'(0) = b \).
(a) Using the time scales \( t_1 = t, \ t_2 = \epsilon t \) and assuming that \( y \sim y_0(t_1, t_2) + \epsilon y_1(t_1, t_2) + \cdots \), show that
\[ y_0 = a \cos(\omega t) + b \sin(\omega t), \]
where \( \omega = 1 + \frac{3}{8} (a^2 + b^2) \epsilon \).
(b) Using the time scales \( t_1 = t, \ t_2 = \epsilon t, \ t_3 = \epsilon^2 t \) and assuming that \( y \sim y_0(t_1, t_2, t_3) + \cdots \), show that
\[ y_0 = a \cos(\omega t) + b \sin(\omega t), \]
where \( \omega = 1 + \frac{3}{8} (a^2 + b^2) \epsilon - \kappa \epsilon^2 \) and \( \kappa = \frac{3}{256} (7a^4 + 46a^2b^2 + 23b^4) \).

3 (Holmes; 2013, 3.20). This problem investigates the subharmonic resonances of Duffing’s equation
\[ y'' + \epsilon \lambda y' + y + \epsilon \kappa y^3 = \epsilon \cos(\Omega t) \]
where \( y(0) = y'(0) = 0 \).
(a) Suppose one attempts to use the regular expansion \( y \sim \epsilon (y_0(t) + \epsilon y_1(t) + \cdots) \). Explain why this is not uniform if \( \Omega = \pm 1 \) (primary resonance), or if \( \Omega = \pm 3 \) (subharmonic resonance). Also explain how your calculation shows that subharmonic resonance is due to the nonlinear interaction of the forcing with the primary resonance mechanism of the system.
(b) Setting \( \Omega = 3 + \epsilon \omega \), find a first-term approximation of the solution that is valid for large \( t \). If you are not able to solve the problem that determines the \( t_2 \) dependence, then find the possible steady states assuming \( \lambda > 0 \).