1. Let $U$ be bounded with a $C^1$ boundary. Prove that there does not exist a bounded linear operator

$$ T : L^p(U) \to L^p(\partial U) $$

for $1 \leq p < \infty$ such that $Tu = u|_{\partial U}$ whenever $u \in C^0(\overline{U}) \cap L^p(U)$. That means, a “typical” function $u \in L^p(U)$ does not have a trace on $\partial U$.

2. Integrate by parts to prove the interpolation inequality:

$$ \|Du\|_{L^2} \lesssim \|u\|_{L^2}^{1/2} \|D^2u\|_{L^2}^{1/2} $$

for all $u \in C^\infty_c(U)$. Assume $U$ is bounded, $\partial U$ is smooth, and prove this inequality if $u \in H^2(U) \cap H^1_0(U)$.

(Hint: Take sequences $\{v_k\}_{k=1}^\infty \subset C^\infty_c(U)$ converging to $u$ in $H^1_0(U)$ and $\{w_k\}_{k=1}^\infty \subset C^\infty(\overline{U})$ converging to $u$ in $H^2(U)$.)

3. Give an example of an open set $U \subset \text{in} \mathbb{R}^n$ and a function $u \in W^{1,\infty}(U)$, such that $u$ is not Lipschitz continuous on $U$.

(Hint: Take $U$ to be the open unit disk in $\mathbb{R}^2$, with a slit removed.)