SECOND ORDER LINEAR HOMOGENEOUS EQUATIONS

Consider \( y'' + p(x)y' + q(x)y = 0 \). \( p, q \) continuous on an interval \( I \).

**Theorem.** The general solution is \( y = c_1y_1 + c_2y_2 \). \( c_1, c_2 \) arbitrary constants.

\( y_1, y_2 \) independent solutions. Called fundamental solutions.

**Definition.** \( y_1 \) and \( y_2 \) are linearly independent on \( I \) if \( c_1y_1 + c_2y_2 = 0 \) on \( I \) implies \( c_1 = c_2 = 0 \).

**Questions.**
1. Why are \( c_1y_1 + c_2y_2 \) solutions?
2. Why are they all solutions?

**Operator Notation**

\[ D = \text{differential operator. Define } L = D^2 + PD + q. \]

The equation is written \( Ly = 0 \).

**Definition.** \( L \) is **linear** if
1. \( L(y_1 + y_2) - Ly_1 + Ly_2 \) for any functions \( y_1, y_2 \)
2. \( L(\alpha y) = \alpha Ly \) for any constant \( \alpha \).

**Example.** \( D \) is linear. \( \therefore D(y_1 + y_2) = (y_1 + y_2)' = y_1' + y_2' = Dy_1 + Dy_2 \).

\[ D(\alpha y) = (\alpha y)' = \alpha y' = \alpha Dy. \]

**Exercise.** \( L = D^2 + PD + q \) is linear.

**Theorem.** (Principle of Superposition.) If \( y_1 \) and \( y_2 \) are solutions of \( Ly = y'' + py' + qy = 0 \), so is \( c_1y_1 + c_2y_2 \).

**Proof.**

\[ L(c_1y_1 + c_2y_2) = L(c_1y_1) + L(c_2y_2) \quad (\because L \text{ is linear}) \]

\[ = c_1Ly_1 + c_2Ly_2 \quad (\because L \text{ is linear}) \]

\[ = 0. \]
Consider \( y'' + p(t)y' + q(t)y = 0 \), \( y(a) = y_0 \), \( y'(a) = y'_0 \), at any point in the interval \( I \).

**Theorem (Existence and Uniqueness)** There is one and only one solution throughout \( I \).

**Proof. Omitted.**

**Theorem.** There is a pair of constants \( c_1 \) and \( c_2 \) so that \( y = c_1 y_1 + c_2 y_2 \) solves the initial value problem.

**Proof.**

\[
\begin{align*}
\begin{bmatrix} y_1(t_0) + c_2 y_2(t_0) = y_0, \\
c_1 y_1'(t_0) + c_2 y_2'(t_0) = y'_0
\end{bmatrix} \quad \text{or} \quad \\
\begin{bmatrix} y_1(t_0) & y_2(t_0) \\
y_1'(t_0) & y_2'(t_0)
\end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y'_0 \end{bmatrix}
\end{align*}
\]

Solvable if

\[
\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.
\]

**Wronskian**

**Definition.**

\[
W(f, g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = (f, g')(t) - (f'g)(t).
\]

If \( f \) and \( g \) are linearly dependent then \( W(f, g) = 0 \).

But the converse is not true. Example: \( f(t) = t^3 \), \( g(t) = t^4 \) on an interval containing 0.
Abel's Theorem. If $y_1$ and $y_2$ are solutions of 
\[ y'' + p(x)y' + q(x)y = 0. \]

Then 
\[ W(y_1, y_2) = Ce^{-\int p(x)\,dx} \]
for some constant $C$.

Proof. 
\[ W' = (y_1y_2' - y_2y_1') = y_1y_2'' - y_1'y_2' = y_1(-p_2y_2 - qy_2') - (p_1y_1'y_2 - p_1'y_1y_2') = -p(y_1y_2' - y_2y_1'). \]

Corollary. Either $W(y_1, y_2)(t) \equiv 0$ or $W$ is never zero.

Let $y_1, y_2$ be solutions of 
\[ y'' + p(x)y' + q(x)y = 0. \]

If $W(y_1, y_2)(t) \equiv 0$ at some $t_0$, then $W(y_1, y_2)(t) \equiv 0$ and $y_1, y_2$ are dependent.

If $W(y_1, y_2)(t) \not\equiv 0$ then $W$ is never zero, and $y_1, y_2$ are independent.

Theorem. If $y_1(t)$ is a solution of 
\[ y'' + p(x)y' + q(x)y = 0, \]
then
\[ y_2(t) = Cy_1(t) \int \frac{e^{-\int p(x)\,dx}}{y_1^2(t)} \, dt. \]
is another solution, independent of $y_1$.

Proof. 
\[ \frac{y_2'}{y_1} = \frac{y_1y_2' - y_2y_1'}{y_1^2} = \frac{W(y_1, y_2)}{y_1^2}. \]

Example. Consider \[ x^2y'' - 3xy' + 4y = 0. \] $a>0$.

Try $y = e^m$. \[ m(m-1) - 3m + 4 = 0, \quad m = 2. \] Therefore $y = e^t$ is a solution.

Note that \[ p(x) = \frac{-2}{x}, \quad e^{-\int p(x)\,dx} = e^{\frac{\ln x}{2} dt} = t^2. \]

From the theorem, 
\[ y_2 = Cy_1 \int \frac{e^{\frac{\ln x}{2}}}{x^2} \, dt = Ct^2. \]

The general solution is 
\[ y = C_1 e^t + C_2 t e^t. \]