LINEAR SECOND-ORDER DEs WITH CONSTANT COEFFICIENTS.

(3) \[ ay'' + by' + cy = 0 \]

\( a, b, c \) (REAL) CONSTANTS. \( a \neq 0 \).

ASSUME: (AND JUSTIFY LATER).

THE GENERAL SOLUTION OF (3) IS \( y = cy_1 + cy_2 \). \( c_1, c_2 \) ARBITRARY CONSTANTS.

\( y_1, y_2 \) INDEPENDENT SOLUTIONS \( \left( y_2 \neq t y_1, y_1 \neq t y_2 \right) \).

\( c_1, c_2 \) ARE CHOSEN TO SATISFY THE INITIAL CONDITIONS.

APPLICATION: SPRING-MASS-DASHPOT SYSTEM

\[ mx'' = -kx - \gamma x' \]

\( x(t) = \) DISPLACEMENT OF MASS FROM EQUILIBRIUM \( (x=0) \)

\( t = \) TIME.

NEWTON’S LAW OF MOTION \( mx'' = F = \) NET FORCE ACTING ON MASS \( m = \) MASS.

\( k = \) SPRING CONSTANT.

\( \gamma = \) DAMPING CONSTANT.

\[ mx'' + \gamma x' + kx = 0 \]

HOW TO SOLVE (3)?

TRY \( y = e^{rt} \), \( r = \) PARAMETER, TO BE DETERMINED.

\( y = re^{rt} \), \( y' = re^{rt} \). (3) \( \Rightarrow a^2e^{rt} + b e^{rt} + ce^{rt} = 0 \).

\[ e^{rt} \Rightarrow ar^2 + br + c = 0 \] CHARACTERISTIC EQUATION.
CASE 1. REAL & DIFFERENT ROOTS \( \lambda_1 \neq \lambda_2 \).

The general solution is
\[ y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \]

Example: \( y'' + 4y' + 3y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \).

The characteristic equation is \( r^2 + 4r + 3 = 0 \), \( r = -3, -1 \).

The general solution is
\[ y = C_1 e^{-3t} + C_2 e^{-t}. \]

Apply the initial conditions: \( C_1 + C_2 = 1 \), \(-3C_1 - C_2 = 0\) \(\Rightarrow C_1 = -1/2, C_2 = 3/2 \).

The solution is
\[ y = -\frac{1}{2} e^{-3t} + \frac{3}{2} e^{-t}. \]

CASE 2. COMPLEX ROOTS \( \lambda = \alpha \pm i\beta \).

The two complex solutions are
\[ y = e^{\alpha t}(C_1\cos \beta t + C_2 \sin \beta t). \]

But prefer to have real-valued solutions.

Theorem: If \( u + iv \) is a complex-valued solution of \( y'' + ay' + by = 0 \), then \( u \) and \( v \) are real-valued solutions.

Proof. \( (u + iv)' + a(u + iv)' + b(u + iv) = 0 \)
\[ (au'' + bu' + cu) + i(au' + bu' + cv) = 0. \]

\( e^{\alpha t}(\cos \beta t + i\sin \beta t) \) by Euler. \(\Rightarrow e^{\alpha t}\cos \beta t \text{ & } e^{\alpha t}\sin \beta t \) are real-valued solutions.

The general solution is
\[ y = e^{\alpha t}(C_1\cos \beta t + C_2 \sin \beta t). \]

Example: \( y'' + 4y' + 3y = 0 \), \( y(0) = 1 \), \( y'(0) = 0 \).

Char. equation is \( r^2 + 4r + 3 = 0 \), \( r = -3, -1 \).

The general solution is
\[ y = e^{-3t}(C_1 \cos t + C_2 \sin t). \]

Initial conditions \( C_1 = 1, C_2 = 0 \).
CASE B. ONE REPEATED ROOT \( \gamma \).

The characteristic equation must be

\[ a(r-\gamma)^2 = 0, \quad ar^2 + 2ar\gamma + a\gamma^2 = 0. \]

The DE must be

\[ ay'' - 2ar\gamma y' + a\gamma^2 y = 0. \]

One solution is \( y_1 = e^{\gamma t} \). Try \( y_2 = y_1 u \), \( u \) a function, to be determined.

\[ y' = r e^{\gamma t} u + e^{\gamma t} u \]
\[ y'' = r^2 e^{\gamma t} u + 2re^{\gamma t} u' + e^{\gamma t} u'' \]

\[ \Rightarrow u \left( ar^2 e^{\gamma t} + 2ar\gamma e^{\gamma t} + a\gamma^2 e^{\gamma t} \right) + u' \left( 2ar e^{\gamma t} - 2ar\gamma e^{\gamma t} + a\gamma^2 e^{\gamma t} \right) = 0. \]

\[ u'' = 0, \quad u = C_1 + C_2 t. \] But just \( u = t \) will be enough.

The general solution is \( y = e^{\gamma t} (C_1 + C_2 t) \).

Example. \( y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0. \)

Character equation \( (r + 2)^2 = 0, \quad r = -2. \)

The general solution is \( y = e^{-2t} (C_1 + C_2 t) \).

Initial conditions \( \Rightarrow C_1 = 1, \quad C_2 = 0. \)

REMARKS ON CASE B.

1. \( e^{(a-i\beta)t} \) will generate two real solutions \( e^{at} \cos \beta t, -e^{at} \sin \beta t \), but it does not generate new solutions.

2. One may write the general solution \( y = c_1 e^{(a+i\beta)t} + c_2 e^{(a-i\beta)t}, \quad c_1, c_2 \) complex.

In order for \( y \) to be real, \( c_1 = \overline{c_2} \) because \( \overline{y} = \overline{c_1 e^{(a+i\beta)t} + c_2 e^{(a-i\beta)t}} \).

The (real) general solution is \( y = \overline{c} e^{(a+i\beta)t} + c e^{(a-i\beta)t} \), \( c \) arbitrary complex.

\[ = C e^{(a+i\beta)t} + c. \]

3. This expression agrees with \( e^{at} (C_1 e^{i\beta t} + C_2 e^{-i\beta t}) \).

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \]

Reading: (b) Section 3.1, 3.3, 3.4.