LINEAR VS. NONLINEAR

EXISTENCE & UNIQUENESS

LINEAR \( y' + py = g(x) \)

- Solution Procedure: Integrating Factor \( \mu = e^{\int p \, dt} \).
  \[
  (\mu y)' = \mu g, \quad y = \frac{\mu C \exp(-\int p \, dt) + \int \mu g \exp(-\int p \, dt) \, dt}{\mu}.
  \]

Consider the associated homogeneous equation \( y' + py = 0 \), separable. \( y = Ce^{-\int pt} = C \mu^t \).

- The general solution is \( y = y_c + yp \). \( y_c = C \mu^t \) is the general solution of \( y' + py = 0 \).

  \( y_p \) is a (particular) solution of \( y' + py = q \).

- Constant Coefficient Case. Consider \( y' + By = B e^{rt} \). \( A + r < 0 \), \( B \) constants.

  (Remark. Integrating Factor \( \mu = e^{\int B \, dt} \)).

  Try \( y_p(t) = Ae^{rt} \) for some constant \( A \) to be determined.

  \( \text{LHS} = A r e^{rt} + B A e^{rt} \), \( \text{RHS} = B e^{rt} \).

  \( A = \frac{B}{B r} \).

NONLINEAR \( y' = f(t, y) \).

- No General Solution Methods.

- A lot of integration, even if solvable.

- Graphical approach if interested in qualitative behavior.

SINGULARITIES = WHERE SOLUTIONS GO BAD?

LINEAR: As long as \( pg \) and \( g(x) \) are continuous and bounded, so are all solutions.

Example. \( y'' + 2y' + y = 0 \), \( y(1) = 2 \).

The solution is \( y(t) = e^t + \frac{1}{e} \).

The solution blows up at \( t = 0 \), where \( p(0) = \frac{2}{e} \) is not defined.

NONLINEAR cannot tell by looking at DE.

Example. \( y'' = -y^2 \), \( y(0) = 1 \).

The solution is \( y(t) = \frac{1}{1-t} \), exists only for \( 0 < t < 1 \).

The solution blows up at \( t = 1 \).

But, the world is nonlinear!