FIRST ORDER LINEAR DEs

\[ a(t) \frac{dy}{dt} + b(t) y = c(t) \]  
LINEAR IN \( y \) \& \( y' \)

STANDARD FORM

\( \left\{ \begin{align*}
    y' + p(t) & = q(t) \\
    y(0) & = y_0
\end{align*} \right. \)

IF \( p(t) \) = CONSTANT:  CONSTANT COEFFICIENT

IF \( y(0) = 0 \), HOMOGENEOUS.

MODELS  * MIXING

+ NEUTRON'S LAW OF COOLING,

+ COMPOUND INTERESTS

+ ROCKET PROPUSION,

EXAMPLE. (MIXING).

\( \begin{array}{c}
Y_i: \text{INFLOW RATE} \\
C_0: \text{INFLOW CONCENTRATION} \\
Y_o: \text{OUTFLOW RATE} \\
C_0: \text{OUTFLOW CONCENTRATION}
\end{array} \)

DETERMINE THE AMOUNT OF SALT AT TIME \( t = \Theta(t) \).

SOLUTION.

\[ \frac{dQ}{dt} = (\text{INFLOW AMOUNT}) - (\text{OUTFLOW AMOUNT}) \]

\[ = C_i Y_i - C_o Y_o \]

\( C_i, Y_i, Y_o \) = GIVEN.  \( C_o = \frac{Q(t)}{V(t)} \),  \( V(t) = \text{VOLUME AT TIME } t \)

\[ = V(0) + \Pi t - Y_o t \]

\[ \frac{dQ}{dt} = C_i Y_i - \frac{Y_o}{V(0) + (\Pi t - Y_o t)} Q \]

IN THE STANDARD FORM

\[ \frac{dQ}{dt} + \frac{Y_o}{V(0) + (\Pi t - Y_o t)} Q = C_i Y_i \]
HOW TO SOLVE \( y' + py = q \)?

IDEA IS TO FIND AN INTEGRATING FACTOR: \( \mu(t) \).

\[(\text{DE}) \times \mu \Rightarrow \mu y' + py \mu = q \mu.\]

WE WANT \( \mu y' + py \mu = (\mu y)' \).

THIS WORKS IF \( \mu' = \mu p \) THE POINT IS, REDUCE A LINEAR EQUATION TO A SEPARABLE.

\[\frac{d\mu}{\mu} = p \, dt, \quad \ln|\mu| = \int p \, dt.\]

\[\mu = e^{\int p \, dt}\]

NO ARBITRARY CONSTANT BECAUSE ONLY \( \mu \) IS ENOUGH.

SUMMARY:

STEP 0. IN THE STANDARD FORM \( y' + py = q \).

STEP 1. CALCULATE \( \mu = e^{\int p \, dt} \).

STEP 2. MULTIPLY BOTH SIDES BY \( \mu \).

STEP 3. INTEGRATE.

EXAMPLE. \( ty' + 2ty = 4t^2 \); \( y(1) = 2 \).

STEP 0. \( y' + \frac{2}{t} y = 4t \).

STEP 1. \( \mu = e^{\int \frac{2}{t} \, dt} = e^{2\ln t} = t^2 \).

STEP 2. \( t^2 y' + 2ty = 4t^3 \). (Check \( t^2 y' + 2ty = (ty^2)' \).)

STEP 3. \( ty^2 = \int 4t^3 \, dt = t^4 + C \).

\( y = \frac{t^2 + C}{t^2} \).

1C. \( y(1) = 1 + C = 2 \). \( C = 1 \).

\[ y = \frac{t^2 + 1}{t^2} \quad (t > 0) \]

EXAMPLE. \( (1 + \cos t) y' - (\sin t) y = 2t \).

DISCUSSED IN CLASS.
CONSTANT COEFFICIENT

EXAMPLE (MIXING, REVISITED)

RECALL \[
\frac{dQ}{dt} + \frac{y_0}{V_0 + (V_0 - y)V_0} \cdot Q = 0 \text{ (1)}
\]

SET \( y = y_0 = V_0 \) CONSTANT. THEN \( V(t) = V \) CONSTANT.

DE REDUCES TO \[
\frac{dQ}{dt} + \sqrt{Q} = V \cdot C(t) \tag{2}
\]

\( u = e^{\sqrt{Q}} \), \( \frac{dQ}{dt} + \sqrt{Q} = V \cdot C(t) \cdot e^{\sqrt{Q}} \)

\[ Q = e^{-\sqrt{Q}} \int \frac{V \cdot C(t) \cdot e^{\sqrt{Q}}}{V} \, dt + e^{-\sqrt{Q}} \cdot C \]

IC: \( Q(0) = Q_0 \)

\[ Q(t) = e^{-\sqrt{Q}} \int_0^t \frac{V \cdot C(t) \cdot e^{\sqrt{Q}}}{V} \, dt + e^{-\sqrt{Q}} \cdot Q_0 \]

STEADY STATE \qquad TRANSIENT SOLUTION

READING: (BD) SECTIONS 2.1 & 2.3