Euler's Numerical Method

Goal: Estimate the solution to \( y' = f(t, y); \quad y(t_0) = y_0 \)

Idea:
- Slope = \( f(t_0, y_0) \)
- Step size = \( h \)

Approximate integral curve by tangent line segments.

Euler's Formula:
- \( t_{n+1} = t_n + h \)
- \( y_{n+1} = y_n + hf(t_n, y_n) \)

Example: \( y' = t^2 - y^2; \quad y(1) = 0, \quad h = 0.1 \)

\[ \begin{array}{c|c|c|c|c|c|c} 
 n & t_n & y_n & f(t_n, y_n) & hf(t_n, y_n) \\
\hline 
0 & 0 & 0 & 1 & 0.1 \\
1 & 1.1 & 0.1 & 1.2 & 0.12 \\
2 & 1.2 & 0.2 & & \\
\end{array} \]

Euler's Method is Rarely Exact.

Common Error Sources:
- Round off errors accumulate.
- Euler's polygon is not a solution.

Too high or too low? Estimating sign of error.

Solution:
- Convex \( (y'' > 0) \) - Too low.
- Concave \( (y'' < 0) \) - Too high.

Remark: The DE is not solvable.
Example (continued) \( y' = t^2 - y^2 \), \( y'' = 2t - 2ty' \) by the chain rule.

At (1,0) \( y'' = 21 - 2 \cdot 0 \cdot (1 - 0) = 2 > 0 \), too low.

The point is: use the DE itself to get info about solutions without solving!

Do better?

1. Use a smaller step size. In general, \( |\text{Error}| \propto h \).

2. Find a better slope.

Example RK2.

\[
slope = \frac{f(t_{n+1/2}, y_{n+1/2})}{2}.
\]

\[\text{Error} \propto h^2 \text{, but more expensive.}\]

Example (pitfall!) \( y' = y^2 \); \( y(0) = 1 \).

Solve by separation of variables. \( y(t) = \frac{1}{1-t} \).

At \( t = 2 \), the actual solution very different from Euler's.

Because computers can't pick the other branch.

Divergent estimates.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( y(1) \times )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>37.6</td>
</tr>
<tr>
<td>0.05</td>
<td>91.25</td>
</tr>
<tr>
<td>0.02</td>
<td>238.21</td>
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</tbody>
</table>

Reading: (BD) Section 2.7.