PRELIMINARIES

**Exponential Function** \( y(t) = e^{kt} \) \( k = \text{constant} \).

**Properties**: (1) \( e^0 = 1 \).
(2) \( e^{kt + c} = e^c e^{kt} \)
(3) \( e^{kt} \) is never zero.
(4) If \( k > 0 \), \( \lim_{t \to \infty} e^{kt} = \infty \), \( \lim_{t \to -\infty} e^{kt} = 0 \).
If \( k < 0 \), \( \lim_{t \to \infty} e^{kt} = 0 \), \( \lim_{t \to -\infty} e^{kt} = \infty \).
(5) For any \( k > 0 \), \( e^{kt} \) grows much faster than any polynomial.

**Examples** \( \lim_{t \to 0} \frac{e^t}{t^3} = \infty \), \( \lim_{t \to \infty} te^{-t} = 0 \).

**Graphs**: \( k > 0 \)

\[
\begin{array}{c}
\text{Graph 1} \quad e^{kt} \\
\text{Graph 2} \quad e^{kt}
\end{array}
\]

You must be completely familiar with these!

**Independent & Dependent Variables**

**Write a function** \( y = x^2 + 2x + 3 \), \( x = \text{independent variable} \), \( y = \text{dependent variable} \).

**A system of equations** \[ \begin{cases} x = t^2 - 1 & t = \text{independent variable} \\ y = 3e^t & x, y = \text{dependent variables} \end{cases} \]

**A function of many variables** \( x = st^2 - s \), \( s, t = \text{independent variables} \), \( x = \text{dependent variable} \).

**A system of many variables** \[ \begin{cases} x = st^2 - s & s, t = \text{independent variables} \\ y = 3e^{st} & x, y = \text{dependent variables} \end{cases} \]

**In ordinary differential equations, one independent & one dependent variable.**
PARAMETERS

For example, \( \int t^2 \, dt = \frac{t^3}{3} + C \)  \( \color{red}{\text{C = parameter. (= constant of integration).}} \)

We say \( y(t) = \frac{t^3}{3} + C \) is a 1-parameter family of functions.

NOTATIONS FOR DERIVATIVES

We write \( \frac{dy}{dt} \) or \( y' \) for the derivative of \( y \) w.r.t. \( t \).

\( \frac{d^2y}{dt^2} \) or \( y'' \) for the 2nd derivative.

\( \frac{d^n y}{dt^n} \) or \( y^{(n)} \) for the \( n \)th derivative.

DIFFERENTIAL EQUATIONS (DEs) RELATE BETWEEN A FUNCTION & ITS DERIVATIVES.

Examples. (1) \( y'' + 4y' + 3y = 0 \).

\( 2) \sqrt{yy''} + (\cos t)e^{y''} + (y' y'')^3 = \sin t \).

The order of a DE is the order of the largest derivative appearing in it.

Examples. (1) is of 2nd order, (2) is of 5th order.

Solving a DE means finding a function that satisfies the equation, called a solution.

Fact of life: For many equations, hard or impossible!

Example 1. (Checking a Solution by Substitution).

\( y' = 3y \) \( y(t) = e^{3t} \) is a solution?

Solution. \( \text{LHS} = 3e^{3t} \), \( \text{RHS} = 3e^{3t} \). \( \therefore \) yes.

Example 2. (Rejecting a Solution by Substitution).

\( y' = y/t \) \( y(t) = t^3 \) is a solution?

Solution. \( \text{LHS} = 3t^2 \), \( \text{RHS} = t^2 \). \( \therefore \) no.
Des usually have more than 1 solutions, involving parameters.

Example. Find all solutions to $y'' = 0$. (This is a calculus problem.)

$y' = c_1$

$y = Ct + c_2$, $c_1, c_2$. Arbitrary constants = parameters.

Initial value problem = differential equation + initial conditions

Example (continued) $y'' = 0$ with $y(1) = 1 \in y'(1) = 2$.

The "general" solution is $y(t) = Ct + c_2$.

$y(1) = c_1 + c_2 = 1$. $y'(1) = c_1 = 2 \Rightarrow c_1 = 2$, $c_2 = -1$. $y(t) = 2t - 1$.

The most important DE $y' = ky$.

In words, rate of change in $y$ or $y'$.

The solution is $y(t) = Ce^{kt}$, $c$ any constant. (Check) Know it by heart!

It models exponential growth (when $k > 0$) or decay (when $k < 0$).

Read: (BD) Sections 1.2 & 1.3.