MATH441 EXAM II

Name ______________________

1. (a) [5] For what value of $k$ is the spring-mass-dashpot system represented by $y'' + 2y' + ky = 0$ critically damped?

   THE CHARACTERISTIC EQUATION IS $r^2 + 2r + k = 0$, $r = -1 \pm \sqrt{1-k}$.

   THE SYSTEM IS CRITICALLY DAMPED IF $1-k = 0$. $k = 1$.

(b) [5] For $k$ greater than the value in part (a), is the system overdamped or underdamped?

   IF $k > 1$, THEN THE EXPRESSION UNDER $\sqrt{}$ BECOMES NEGATIVE,
   AND ROOTS OF THE CHARACTERISTIC EQUATION BECOME NON-REAL. UNDERDAMPED

(c) [10] Suppose a solution of $y'' + 2y' + ky = 0$ vanishes at $t = 1$ and then again at $t = -2$ but not in between. What is $k$?

   THE SYSTEM IS UNDERDAMPED.

   THE GENERAL SOLUTION IS $y = e^{-t}(c_1\cos\omega_1t + c_2\sin\omega_1t)$, $\omega_1 = \sqrt{1-k}$. (SEE (a)).

   $y =$

   $t = 1$

   $t = 2.$

   PERIOD $= \frac{\omega_1}{\pi} = 2$.

   $\omega_1 = \pi.$

   $k = \pi^2 - 1$. 

   $y =$

   $t = 1$

   $t = 2.$
2. (a) [10] If $\frac{1}{2} t \sin(2t)$ is a solution of $my'' + \gamma y' + ky = 4 \cos(2t)$, determine $m$, $\gamma$ and $k$.

The system is resonant: $p(r) = mr^2 + \gamma r + k = m(\gamma^2 + 4)$ \quad \therefore \gamma = 0, \quad \phi = 4 \pi r.

\[ p'(r) = 2mr. \]

\[ y_p = \text{Re} \left( \frac{4t e^{2it}}{p'(2t)} \right) = \text{Re} \left( \frac{4t e^{2it}}{4m} \right) = \frac{t}{m} \sin(at) = \frac{1}{2} \sin(2t). \quad \therefore m = 2. \]

\[ \therefore m = 2, \quad \gamma = 0, \quad \phi = 8 \]

(b) [10] At what $\omega$ does the sinusoidal solution to $y'' + 2y' + 10y = \cos(\omega t)$ have the maximal amplitude? (For full credit, make a relevant calculation, rather than using a formula.)

Consider $\ddot{z} + 2\dot{z} + 10z = \cos(\omega t)$.

\[ p(\omega) = (\omega^2 + 2\omega + 10 = 10 - \omega^2 + 2\omega i 
\]

\[ \therefore z = \frac{e^{i\omega t}}{p'(\omega)} = \frac{1}{|p'(\omega)|} e^{-i\phi} e^{i\omega t}. \quad \phi = \arg(p(\omega)). \]

\[ y_p = \frac{1}{|p'(\omega)|} \cos(\omega t - \phi) \quad \therefore \text{Amplitude} = \frac{1}{|p'(\omega)|} \]

\[ \text{Minimize} |p'(\omega)|^2 = (10 - \omega^2 + \omega^2)^2 = (\omega^2 - 10)^2 + 100 = (\omega^2 - 8)^2 + 36. \quad \therefore \omega = 2\sqrt{2}. \]

(c) [5] At what $\omega$ is the phase lag of the sinusoidal solution to $y'' + 2y' + 10y = \cos(\omega t)$ equal to $\pi/2$?

\[ \text{Phase lag} = \arg(p(\omega)) = \frac{\pi}{2}. \]

\[ \tan (p(\omega)) = \frac{2\omega}{10 - \omega^2} = \tan \frac{\pi}{2} = \infty. \]

\[ 10 - \omega^2 = 0. \quad \therefore \omega = \sqrt{10}. \]
3. Let \( L[y] = y'' + p(t)y' + q(t)y \), where \( p \) and \( q \) are continuous on an open interval \( I \).

(a) [10] If \( L[y_1] = L[y_2] = 0 \) on \( I \) and if \( y_1 \) and \( y_2 \) have maxima or minima at the same point in \( I \), show that \( y_1 \) and \( y_2 \) are linearly dependent.

\[
\text{NOTE THAT } \quad \dot{y_1}(t_0) = \dot{y_2}(t_0) = 0, \text{ WHERE } t_0 = \text{MAX OR MIN POINT.}
\]

\[
W(y_1, y_2)(t_0) = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}_{t=t_0} = 0.
\]

But by Abel's Theorem, \( W(t) = C \cdot e^{\int p(t) dt} = 0 \) at \( t = t_0 \).

Therefore \( C = 0 \), and \( W(t) = 0 \) for all \( t \in I \).

\(. y_1 \text{ AND } y_2 \text{ ARE LINEARLY DEPENDENT.}\)

(b) [10] If \( L[t] = g(t) \), \( L[t + e^t] = g(t) \) and \( L[1 + t + e^t] = g(t) \), find the general solution of \( L[y] = g(t) \).

\[
L[te^t] = L[t + e^t - t] = L[t + e^t] - L[t] = 0.
\]

\[
L[1 + t + e^t] = L[1 + t + (t + e^t)] = 0.
\]

Therefore \( t, e^t \) are independent solutions of \( L[y] = 0 \). \( y_c = C_1 + C_2 e^t \).

\( y = y_c + y_p = \boxed{C_1 + C_2 e^t + t} \)
4. (a) [15] Find a solution of \( y'' + 2y' + 5y = 4e^{-t} \cos(2t) \).

Consider \( r^2 + 2r + 5 = 4e^{(-1+2i)t} \).

\[ p(0) = r^2 + 2r + 5 = 0, \quad r = -1 \pm 2i. \]

\[ p'(0) = 2r + 2. \]

\( -1 \pm 2i \) is a simple root of \( p(t) \).

\[ y_p = \text{Re} \left( \frac{4te^{(1+2i)t}}{p'(1+2i)} \right) = \text{Re} \left( \frac{d}{dt} \frac{4te^{(1+2i)t}}{2(1+2i)+2} \right) = \text{Re} \left( \frac{4t e^{-t} e^{2i t}}{4i} \right) = \boxed{te^{-t} \sin(2t)} \]

(b) [15] Find a solution of \( y'' + y' + y = 2te^t \).

\[ p(0) = 1 + r + 1. \quad p(0) = 3 \neq 0. \]

Try \( y = e^t (At + B) \equiv e^t u(t) \).

\[ p(D)(e^{t}u(t)) = e^t p(D+1)u(t) \quad \text{by Exponential Shift Law}. \]

\[ p(D+1)u = (D^2 + D + 3)u = 2t \]

\[ u = A + B \quad \therefore A = \frac{2}{3}, \quad B = -\frac{2}{3}. \]

\[ u' = A \]

\[ u'' = 0. \]

\[ \therefore y_p = \frac{2}{3} e^t (t - 1) \]
5. (a) [12] Given that \( y_1(t) = t \) is a solution to \( t^2y'' - t(t + 2)y' + (t + 2)y = 0 \), find the general solutions on the interval \( t > 0 \).

\[
\begin{align*}
W &= C \cdot \exp \int \frac{t + 2}{t} \, dt = t^2e^t \\
\left( \frac{y_2}{y_1} \right)' &= \frac{W}{y_1^2} \\
y_2 &= y_1 \int \frac{W}{y_1^2} \, dt = t \int \frac{Ct^2e^t}{t^2} \, dt = Cte^t.
\end{align*}
\]

\[
\text{Therefore, the general solution is } y_0 = C_1t + C_2te^t.
\]

(b) [13] Find a particular solution of \( t^2y'' - t(t + 2)y' + (t + 2)y = t^3 \), using the variation of parameters formula: If \( y_1 \) and \( y_2 \) are independent solutions of \( y'' + p(t)y' + q(t)y = 0 \) then

\[
y_p = y_1 \int \frac{-y_2g}{W(y_1, y_2)} \, dt + y_2 \int \frac{y_1g}{W(y_1, y_2)} \, dt.
\]

is a particular solution of \( y'' + p(t)y' + q(t)y = g(t) \).

\[
\begin{align*}
\text{In the standard form: } & \quad y'' - \frac{te^t}{t^2}y' + \frac{te^t}{t^2}y = t^3 \\
y_1 &= t, \quad y_2 = te^t \\
W &= \begin{vmatrix} t & te^t \\ 1 & te^t + te^t \end{vmatrix} = te^t.
\end{align*}
\]

\[
y_p = t \int \frac{-tte^t}{t^2e^t} \, dt + te^t \int \frac{tte^t}{t^2e^t} \, dt
\]

\[
= -t^2 - te^t e^{-t}
\]

\[
= -t^2 - \frac{te^t}{y_1}
\]

\[
\text{Better yet, } y_p = -t^2
\]