1. (a) [5] In a perfect environment, the population of Norway rat that breed on the U of I campus increases by a factor of $e \approx 2.7182818284...$ each year. (That is to say, $y(1) = ey(0).$) Model this exponential growth by a differential equation. What is the growth rate $k$?

Let $y(t) =$ # of rats at time $t$ (in years)

Natural growth: \[ \frac{dy}{dt} = ky \]

The solution is $y(t) = y(0)e^{kt}$.

$y(t) = y(0)e^{kt} = ey(0) = e^{k-1}$

(b) [5] The university is a limited environment, where the Norway rat population has a saturation level. Write down the logistic equation modeling this. (You may use "k" for the natural growth rate here if you failed to find it in (a).)

Logistic equation: \[ \frac{dy}{dt} = ky - ay^2 \]

$k = 1$, $a > 0$ constant.

(c) [5] Sketch some solutions for the equation in part (b). Mark all critical points. Label each critical point as stable or unstable.

[5] If the saturation level is 1000 rats, determine the logistic equation. (The saturation level is the upper bound of the population that is approached, but not exceeded by growing populations starting below this value.)

Saturation level $= \frac{1}{a} = 1000$.

The equation is \[ y' = y - ay^2/1000 \].
(d) [10] The university pest control service intends to control these rats by killing them at a constant rate of \( r \) rats per year. If it wants to limit the rat population to 75% of the saturation level, what rate \( r \) it should aim for (in rats per year)?

Logistic Equation with Harvesting: \( y' = y - y^2/1000 - r \).

Wanted the critical point at \( y = 0.75(1000) = 750 \).

\[
0 = 750 - (750)^2/1000 - r
\]

\[
\therefore r = 67.5.
\]

2. (a) [10] Estimate \( y(2.2) \), where \( y \) is the solution of the differential equation \( dy/dx = y^2 - x^2 \) with \( y(2) = 0 \) using Euler’s method with step size 0.1.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
<th>( y_n )</th>
<th>( y_n^2 - x_n )</th>
<th>( 0.1(y_n^2 - x_n^2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-4.0</td>
<td>-0.4</td>
</tr>
<tr>
<td>1</td>
<td>2.1</td>
<td>-0.4</td>
<td>-4.25</td>
<td>-0.425</td>
</tr>
<tr>
<td>2</td>
<td>2.2</td>
<td>-0.825</td>
<td>-0.825</td>
<td>-0.0825</td>
</tr>
</tbody>
</table>
(b) [10] Sketch a direction field for the differential equation \( \frac{dy}{dx} = y^2 - x^2 \) using isoclines for slopes 0, ±1, ±2.

[5] On the diagram, sketch the graph of the solution of the equation with \( y(2) = 0 \).

\[
\text{Slope at (2,0)} = \frac{0^2 - 2^2}{2} = -4.
\]

(d) [5] A certain solution \( y \) of the equation in part (a) has a local extremum at \( x = -1 \). Is the extremum a maximum or a minimum? For full credit, make a relevant calculation, rather than merely relying on the picture.

\[
y' = 2yy' - 2x = 2 > 0
\]
\[
\therefore y(-1) \text{ is a maximum.}
\]

3. (a) [10] Find the solution of \( t \frac{dy}{dt} + y = \cos t \) such that \( y(\pi) = 1 \).

\[
(ty)' = \cos t
\]
\[
ty = \sin t + C
\]
\[
y = \frac{\sin t + C}{t}
\]
\[
y(\pi) = \frac{0 + C}{\pi} = 1.
\]
\[
C = \pi.
\]
\[
\therefore y = \frac{\sin t + \pi}{t}
\]
(b) [10] Find all solutions of \((xy + x)dx = ydy\).
\[
\begin{align*}
&x(y+1)dx = y\,dy \\
&x\,dx = \frac{y}{y+1}\,dy \quad \text{if} \quad y+1 \neq 0 \\
&x\,dx = \left(1 - \frac{1}{y+1}\right)\,dy \\
&\frac{1}{2}x^2 + C = y - \ln|y+1| \\
&y - \ln|y+1| = \frac{1}{2}x^2 + C, \quad y = -1.
\end{align*}
\]

4. (a) [5] Transform the initial value problem \(y' = y, y(0) = 1\) into an equivalent integral equation.
\[
y(t) = 1 + \int_0^t y(s)\,ds
\]

(b) [5] Compute Picard iterates \(\phi_n(t)\) for the initial value problem \(y' = y, y(0) = 1\).
\[
\begin{align*}
\phi_0(t) &= 1, \\
\phi_1(t) &= 1 + \int_0^t 1\,ds = 1 + t, \\
\phi_2(t) &= 1 + \int_0^t (1 + s)\,ds = 1 + t + \frac{t^2}{2}, \\
&\vdots \\
\phi_n(t) &= 1 + t + \frac{t^2}{2} + \ldots + \frac{t^n}{n!}.
\end{align*}
\]
(c) [15] What is the largest interval of existence that the Picard existence theorem predicts for the solution of the initial value problem $dy/dt = t^2 + y^2$, $y(0) = 0$?

\[
\begin{align*}
\frac{d}{dt}(t^2 + y^2) &= 2t + 2y, \\
\max_{t \in T} (t^2 + y^2) &= T^2 + K^2, \\
\min_{y \in K} \frac{K}{T^2 + K^2} &\leq \frac{1}{a_T} \quad \text{for all } T, K > 0, \\
T \leq \min\left(T, \frac{K}{T^2 + K^2}\right) &\leq \min\left(T, \frac{1}{2T}\right) \leq \frac{1}{4}
\end{align*}
\]

(d) [15] Find all nonzero solutions of the initial value problem $dy/dt = y^{1/2}$, $y(0) = 0$. Does this violate the uniqueness theorem? Explain.

\[
\begin{align*}
y^{1/2} dy &= dt \quad \text{if } y \neq 0, \\
2y^{3/2} &= t + C, \\
y &= \frac{1}{4}(t + C)^2. \\
\end{align*}
\]

\[\text{Note that } \frac{d}{dt}(y^{1/2}) = \frac{1}{2} (t + C)^{-1/2} \quad \text{for } t \geq 0 \]

This does not violate the uniqueness theorem because \( \frac{d}{dt}(y^{1/2}) = \frac{1}{2} y^{-1/2} \) is not even defined at \( y = 0 \).

And hence the uniqueness theorem does not apply.