MATH441 Guide and Practice Exam II

Study guide

1. Models. A second-order linear differential equation is of the form \( y'' + p(t)y' + q(t)y = g(t) \). In the operator notation
\[
(D^2 + p(t)D + q(t))y = g(t).
\]
In case the coefficients are constant the equation can be written in terms of the characteristic polynomial \( P(r) = ar^2 + br + c \) as \( P(D)y = g(t) \).

Spring-mass-dashpot system: \( P(r) = mr^2 + \gamma r + k \).

2. Homogeneous equations. The general solution of \( (D^2 + p(t)D + q(t))y = 0 \) is \( y = c_1y_1 + c_2y_2 \), where \( y_1, y_2 \) are linearly independent and \( c_1, c_2 \) are arbitrary constants. The “mode” \( e^{rt} \) solves \( P(D)y = 0 \) exactly when \( P(r) = 0 \); if the coefficients are real and \( r = \alpha + i\beta \) with \( \beta \neq 0 \) then \( e^{\alpha t} \cos(\beta t) \) and \( e^{\alpha t} \sin(\beta t) \) are independent real solutions; if \( r \) is a repeated root one needs \( te^{rt} \) also. If all roots have negative real part then all solutions decay to zero as \( t \to \infty \) and the equation is called stable.

Spring-mass-dashpot system: the equation is overdamped if the roots are real and distinct \( (k < \gamma^2/4m) \), critically damped if there is only one (repeated) root \( (k = \gamma^2/4m) \) and underdamped if the roots are not real \( (k > \gamma^2/4m) \), undamped if the roots are purely imaginary \( (\gamma = 0) \). In the undamped case the general solution is \( Ae^{\alpha t} \cos(\omega_0t - \phi) \), where \( \omega_0 = \sqrt{k/m} \) is the natural frequency, and in the underdamped case the general solution is \( Ae^{-\gamma t/2m} \cos(\omega_1t - \phi) \), where \( \omega_1 = \sqrt{k/m - (\gamma/2m)^2} \) is the quasi frequency.

3. The Wronskian. \( W(y_1, y_2)(t) = (y_1y_2' - y_1'y_2)(t) \).

Abel’s Theorem: If \( y_1 \) and \( y_2 \) are solutions of \( (D^2 + p(t)D + q(t))y = 0 \) then \( W(y_1, y_2)(t) = c\exp(-\int p(t)dt) \), where \( c \) is a certain constant. Consequence: \( W(y_1, y_2)(t) \) either is zero for all \( t \) or else is never zero.

Applications: If \( y_1 \) is a known non vanishing solution of \( (D^2 + p(t)D + q(t))y = 0 \) then a second solution satisfies \( (y_2/y_1)' = W(y_1, y_2)/y_1^2 \).

4. Inhomogeneous equations. The general solution of \( (D^2 + p(t)D + q(t))y = g(t) \) is \( y = y_p + y_c \), where \( y_p \) is a particular solution of \( (D^2 + p(t)D + q(t))y = g(t) \) and \( y_c \) is the general solution of \( (D^2 + p(t)D + q(t))y = 0 \).

5. Exponential response formula. If \( P(\alpha) \neq 0 \) then \( e^{\alpha t}/P(\alpha) \) solves \( P(D)y = e^{\alpha t} \). If \( P(\alpha) = 0 \) but \( P'(\alpha) \neq 0 \) then \( te^{\alpha t}/P'(\alpha) \) solves \( P(D)y = e^{\alpha t} \). (Etc.)

6. Complex replacement. If \( P(r) \) has real coefficients then solutions of \( P(D)y = e^{\alpha t} \cos(\beta t) \) are real parts of solutions of \( P(D)z = e^{(\alpha+i\beta)t} \).

7. Undetermined coefficients. \( P(D)y = (\text{polynomial of degree } m) \) has exactly one polynomial, which has degree at most \( m \).

8. Variation of parameter. \( y_p = -y_1 \int \frac{y_2g}{W(y_1, y_2)} + y_2 \int \frac{y_1g}{W(y_1, y_2)} \).

9. Frequency response. \( g(\omega)\cos(\omega t - \phi) \) solves \( P(D)y = \cos(\omega t) \), where \( g = |p(i\omega)|^{-1} \) is the gain and \( \phi = \arg(p(i\omega)) \) is the phase lag.
Practice Exam

1. For (a)-(b), the mass and spring constant in a certain spring-mass-dashpot system are known – $m = 1$, $k = 25$ – but the damping constant $\gamma$ is not known. It’s observed that for a certain solution $y(t)$ of $y'' + \gamma y' + 25y = 0$, $y(\pi/6) = 0$ and $y(\pi/2) = 0$, but $y(t) > 0$ for $\pi/6 < t < \pi/2$.
   (a) Is the system underdamped, critically damped, or overly damped?
   (b) Determine the value of $\gamma$.

Problems (c)-(d) concerns the sinusoidal solution $y(t)$ of $y'' + 4y' + 9y = \cos(\omega t)$.
   (c) For what value of $\omega$ is the amplitude of $y(t)$ maximal?
   (d) For what value of $\omega$ is the phase lag exactly $\pi/4$?

2. For (a)-(b), assume that $p$ and $q$ are continuous on an open interval $I$ and consider $y'' + p(t)y' + q(t)y = 0$ on $I$.
   (a) If the graph of a solution is tangent to the $t$ axis at some point, show that it must be identically zero.
   (b) If $y_1$ and $y_2$ are solutions with a common inflection point $t_0$ on $I$, show that they cannot be independent unless $p(t_0) = q(t_0) = 0$.

For (c)-(d), assume that $\cos t$ and $t$ are both solutions of the equation $P(D)y = g(t)$, for a certain polynomial $P(r)$ and a certain function $g(t)$.
   (a) Write down a nonzero solution of the equation $P(D)y = 0$.
   (b) Write down a solution $y(t)$ of $P(D)y = g(t)$ such that $y(0) = 2$.

3. (a) Find the general solution of $y'' + 4y = 3 \sin 2t$.
   (b) Find a solution of $y'' + 2y' + y = te^{-t} \cos t$.

4. (a) Find the general solution of Bessel’s equation (of order 1/2) $x^2 y'' + xy' + (x^2 - 1/4)y = 0$, $x > 0$, given that $y_1 = \sin(x)/\sqrt{x}$ is a solution.
   (b) Find a particular solution of $x^2 y'' + xy' + (x^2 - 1/4)y = x^{3/2} \cos x$. 
Solutions.

1. (a) underdamped.
   (b) \( \gamma = 8 \).
   (c) \( \omega = 1 \).
   (d) \( \omega = \sqrt{13} - 2 \).

2. (c) \( c(\cos t - t) \).
   (d) \( y = 2 \cos t - t \).

3. (a) \( y = c_1 \cos 2t + c_2 \sin 2t - (3/4)t \cos 2t \).
   (b) \( y = e^{-t}(-t \cos t + 2 \sin t) \).

4. (a) \( c_1 \cos(x)/\sqrt{x} + c_2 \sin(x)/\sqrt{x} \).
   (b) \( (1/2)\sqrt{x} \sin x \).