MATH441 Guide and Practice Exam I

Study guide

1. Direction fields. (This part is not in the textbook.) Sketch the direction field of the DE \( y' = f(x, y) \) using the isocline method and sketch integral curves. Use properties of integral curves to obtain qualitative information of solutions.

2. Numerical approximation: Euler’s method. (Boyce and DiPrima, Section 2.7.) Approximate \( y(t_{n+1}) \) of the initial value problem \( y' = f(t, y), \ y(t_0) = y_0 \), by evaluating

\[
  t_{n+1} = t_n + h, \quad y_{n+1} = y_n + hf(t_n, y_n),
\]

where \( h \) is the step size. Use the second derivative to determine if Euler’s approximation is too high or too low.

3. Separable equations. (Boyce and DiPrima, Section 2.2.) Separate the variables and solve the DE \( \frac{dy}{dx} = f(x)g(y) \). Lost solutions?

4. Linear equations. (Boyce and DiPrima, Sections 2.1. and 2.3.) A first-order linear DE takes the form \( y' + p(t)y = g(t) \). Find the integrating factor \( \mu = \exp \left( \int p \, dt \right) \) to solve. In case \( p(t) = k > 0 \) a constant, identify the transient solution and the steady state.

Study the mixing model and Newton’s law of cooling.

Use substitution to solve a Bernoulli’s equation \( y'(t) + p(t)y = q(t)y^n \) and a homogeneous equation \( \frac{dy}{dx} = f(y/x) \).

5. Autonomous equations and population dynamics. (Boyce and DiPrima, Section 2.5.) For an autonomous equation \( y' = f(y) \), find critical points, sketch several graphs of solutions and determine the stability and instability of each critical point.

Study the exponential growth model \( y' = ay \), the logistic model \( y' = (a - by)y \) and the logistic model with harvesting \( y' = (a - by)y - h \) using direction fields. Make interpretations.

6. Existence and uniqueness. (Boyce and DiPrima, Section 2.8.) If \( f(t, y) \) and \( \partial f/\partial y(t, y) \) are continuous in \( |t - t_0| < T \) and \( |y - y_0| < K \) then the initial value problem \( y' = f(t, y), \ y(t_0) = y_0 \) has a unique solution in \( |t - t_0| < T_1 = \min(T, K/M) \), where \( |f(t, y)| \leq M \).

The proof is to transform the IVP to the integral equation \( y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) \, ds \) and uses Picard’s iteration

\[
  y_0(t) = y_0, \quad y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s)) \, ds.
\]

7. Linear vs. nonlinear. (Boyce and DiPrima, Section 2.4.) The general solution of the linear equation \( y' + p(t)y = g(t) \) takes the form \( y = y_c + y_p \), where \( y_c \) is the general solution of \( y' + p(t)y = 0 \). Discuss two nonlinear equations \( y' = y^2 \) and \( y' = y^{1/3} \).

8. Complex numbers. (This part is not in the textbook.) Solve \( y' + ky = \cos(\omega t) \) by complexifying the equation and using Euler’s formula.
Practice Exam

1. A certain computer chip sheds heat at a rate proportional to the difference between its temperature and that of its environment.
   (a) Write down a differential equation controlling the temperature of the chip, as a function of time measured in minutes, if the temperature in the environment is a constant 20°C. Your equation will have a constant in it which can’t be determined from the data given so far.
   (b) What is the general solution of this equation? (This will still involve the unknown constant.)
   (c) It is observed that if the chip is powered down at \( t = 0 \) at a temperature of 70°C in a room at 20°C, its temperature at \( t = 10 \) minutes is 60°C. Use this new information to complete the determination of the differential equation.
   (d) Sketch the graphs of some solutions to the autonomous equation \( \frac{dy}{dt} = 2y - 3y^2 + y^3 \). Be sure to include at least one solution with values in each interval above, below and between critical points. Label each critical point as stable or unstable.

2. Let \( y = f(x) \) be the solution of \( \frac{dy}{dx} = x^2 - y^2 \) with \( f(-2) = 0 \).
   (a) Sketch the isoclines for slopes \(-2, 0, 2\). Sketch the direction field along them.
   (b) On the same diagram, sketch the graph of the solution \( f(x) \). What is its slope at \( x = -2 \)?
   (c) Suppose that the function \( f(x) \) reaches a local maximum at \( x = x_0 \). What is \( f(x_0) \)?
   (d) Use two steps of Euler’s method to estimate \( f(-1) \). The estimate is too high or too low?

3. (a) Find the general solution of \( t\frac{dy}{dt} + 2y = t^2 \).
   (b) Find an oscillating solution to the differential equation \( \frac{dy}{dt} + 2y = \cos(2t) \). Express your answer in the form \( A\cos(\omega t - \phi) \). You may use any method to find this solution.
   (c) Find all solutions of \( (x^2y + xy - y)dx + (y - 2)dy = 0 \).

4. (a) Transform the initial value problem \( y' = 2t(1 + y), \ y(0) = 0 \) into an equivalent integral equation.
   (b) Compute Picard iterates \( \phi_n(t) \) for \( n = 0, 1, 2 \) for the initial value problem in part (b).
   (c) What is the largest interval of existence that the existence theorem predicts for the solution of the initial value problem \( \frac{dy}{dt} = 1 + y^2, \ y(0) = 0? \)
   (d) Find a solution of the initial value problem \( \frac{dy}{dt} = t\sqrt{1 - y^2}, \ y(0) = 1 \) other than \( y(t) = 1 \). Does this violate the uniqueness theorem? Explain.
Solutions.

1. (a) Let $y(t)$ be the temperature of the chip in degrees C. $y' = k(20 - y)$.
   (b) $y = 20 + Ce^{-kt}$.
   (c) $k = (\log(50) - \log(40))/10$.
   (d) $y = 0, 2$ unstable; $y = 1$ stable.

2. (b) The slope at $(-2,0)$ is 4.
   (c) $f(x_0) = -x_0$.
   (d) 1.125, too high.

3. (a) $y = t^2/4 + C/t^2$.
   (b) $y = (\sqrt{2}/4)\cos(2t - \pi/4)$.
   (c) $y - \ln(y^2 + \frac{1}{2}x^3 + \frac{1}{2}x^2 - x + C = 0$.

4. (a) $y = \int_0^t 2s(1 + y(s))ds$.
   (b) $\phi(t)_0 = 0$, $\phi_1(t) = t^2$, $\phi_2(t) = t^2 + t^4/2$.
   (c) $|t| \leq 1/2$.
   (d) $y(t) = \sin(t^2/2 + \pi/2)$. 